iii. *The kernel $K(x, \xi)$ is normalized, that is*

$$K(x, x^+) - K(x, x^-) = 1 \; ; \; K_x(x, x^+) - K_x(x, x^-) = 0.$$

iv. *The kernel $K_x(x, \xi)$, where*

$$K_x(x, x) \equiv K_x(x, x^+) = K_x(x, x^-),$$

is possessed of a reciprocal, $\mathfrak{E}(x, \xi)$.*

v. *The constants a and b defined by*

(14)
$$a = 1 - K(0, 0^+) + \int_0^1 K(0, \xi) \, \mathfrak{E}(\xi, 0) d\xi,$$

$$b = K(0, 1) - \int_0^1 K(0, \xi) \, \mathfrak{E}(\xi, 1) d\xi,$$

are different from zero.

Under these hypotheses we have the following conclusions:

(I) *The integral equation* (2) *has infinitely many characteristic values. Let*

(15)
$$\rho_k' = \log(-b/a) + 2k\pi i \qquad\qquad (k = 0, \pm 1, \pm 2, \cdots)$$

be the characteristic values of the differential boundary problem

($\bigstar\bigstar$)
$$u'(x) + \rho u(x) = 0 \; ; \; au(0) + bu(1) = 0.$$

Then, if δ is any positive number arbitrarily small but fixed, an R_δ is available which is so large that, outside the circle $|\rho| = R_\delta$, all the characteristic values of (2) *lie in the interiors of the circles of radius δ around the points* (15), *each circle containing one and only one characteristic value of* (2).

(II) *Outside the circle $|\rho| = R_\delta$ all the characteristic values of* (2) *are simple poles of the resolvent kernel $\mathfrak{R}(x, \xi, \rho)$ and to each of them there corresponds a single pair $u(x)$, $v(x)$ of the fundamental solutions of* (2) *and of the associated integral equation* (13).

(III) *Let*

(16)
$$u_1(x), v_1(x), u_2(x), v_2(x), \cdots, u_r(x), v_r(x), \cdots$$

be the complete biorthogonal set of the fundamental functions of (2), (13) *and*

$$u_k^{(1)}(x), v_k^{(1)}(x) \qquad\qquad (k = 0, \pm 1, \pm 2, \cdots)$$

the complete biorthogonal set of the fundamental solutions of ($\bigstar\bigstar$) *and of the adjoint problem*

($\bigstar\bigstar'$)
$$v'(x) - \rho v(x) = 0 \; ; \; bv(0) + av(1) = 0.$$

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*This condition is somewhat less general than the corresponding Langer's condition (v) (L, p. 592). We expect to discuss in another paper the case where $K_x(x, \xi)$ has no reciprocal.

Let $f(x)$ be any function integrable on $(0, 1)$. If we set

$$(17) \quad \sum(f) \equiv \sum_{\nu=1}^{\infty} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt \; ; \; \sum_N(f) \equiv \sum_{\nu=1}^{N} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt,$$

$$(18) \quad S(f) \equiv \sum_{\nu=-\infty}^{\infty} u_\nu^{(1)}(x) \int_0^1 f(t)v_\nu^{(1)}(t)dt; \; S_N(f) \equiv \sum_{\nu=-N}^{N} u_\nu^{(1)}(x) \int_0^1 f(t)u_\nu^{(1)}(t)dt,$$

the series $\sum(f)$ and $S(f)$ are equiconvergent on the interior of $(0, 1)$, that is

$$(19) \qquad\qquad \sum_N(f) - S_N(f) \to 0 \; as \; N \to \infty,$$

uniformly on every interval interior to $(0, 1)$. At the end points $0, 1$ the difference (19) tends respectively to

$$(20) \qquad\qquad C_0 \int_0^1 f(t)\alpha(t)dt, \qquad C_1 \int_0^1 f(t)\alpha(t)dt$$

where

$$(21) \qquad\qquad \alpha(\xi) = K_\xi(0,\xi) - \int_0^1 K(0,s)\,\mathfrak{E}_\xi(s,\xi)ds,$$

and C_0, C_1 are constant factors which depend only on $K(x, \xi)$ and do not depend on $f(x)$.

(IV) *No modifications are necessary in the statements* (I) *and* (II) *if the differential problem* ($\bigstar\bigstar$) *is replaced by*

$$(\bigstar\bigstar\bigstar) \qquad u'(x) + \rho u(x) = 0 \; ; \; au(0) + bu(1) = \int_0^1 \alpha(\xi)u(\xi)d\xi.$$

Let $G(x, t, \rho)$ be the Green's function of the problem ($\bigstar\bigstar\bigstar$) and

$$(22) \qquad\qquad \rho_1'', \, \rho_2'', \, \cdots, \rho_\nu'', \, \cdots$$

the set of the characteristic values of ($\bigstar\bigstar\bigstar$). Let

$$(C_R) \qquad\qquad |\rho| = R$$

be a circle around the origin, which does not pass through any of the points ρ_ν, ρ_ν', ρ_ν''. If $\rho_1, \rho_2, \cdots, \rho_N$ are the characteristic values of (2) within (C_R), then

$$(23) \qquad \lim_{R \to \infty} \left\{ \sum_N(f) - \frac{1}{2\pi i} \int_{(C_R)} d\rho \int_0^1 G(x,t,\rho)f(t)dt \right\} = 0,$$

uniformly on $(0, 1)$.

34. The second part of the statement (IV) follows immediately from Theorem 4 since the integro-differential problem ($\bigstar$) satisfies the conditions (A), (B), (C) and the Green's function $\Gamma(x, t, \rho)$ of this problem coincides with the resolvent kernel $\Re(x, t, \rho)$.

In order to prove the statements (I), (II) and the first part of (IV) let ρ_0' be any pole of $G'(x, t, \rho)$ outside $|\rho| = R_s$. If R_s is sufficiently large, then to each ρ_0' there corresponds one and only one pole ρ_0'' of ($\bigstar\bigstar\bigstar$) which is within the circle (c) of radius δ around ρ_0' and vice versa. Let $G'(x, t, \rho)$ be the Green's function of ($\bigstar\bigstar$). Take the integrals

$$J_c(\Gamma), \ J_c(G'), \ J_c(G)$$

which have been used already in § 26. The principal part of $G'(x, t, \rho)$ corresponding to the pole $\rho = \rho_0'$ is

$$U_0(x)V_0(t)/(\rho - \rho_0') \ ; \ \int_0^1 U_0(x)V_0(x)dx = 1,$$

where $U_0(x)$, $V_0(x)$ are respectively the fundamental solutions of the problem ($\bigstar\bigstar$) and of the adjoint problem ($\bigstar\bigstar'$) for $\rho = \rho_0'$ (D, 15). Hence

$$J_c(G') = 1.$$

On the other hand, the same argument being applied to $\Gamma(x, t, \rho) \equiv \Re(x, t, \rho)$ shows

$$J_c(\Gamma) = \sigma$$

where σ denotes the total number of pairs u_ν, v_ν in (16) which correspond to the characteristic values of (2) within (c). We have, however,

$$\sigma - 1 = J_c(\Gamma - G') = J_c(\Gamma - G) + J_c(G - G') = O(1/r),$$

where r is the shortest distance from the origin to the contour of (c) (§ 26). Hence $\sigma = 1$, provided R_s is sufficiently large. Thus the statements (I), (II), (IV) are proved.

The statement (III) follows immediately from Theorem 6; we observe that in the present case

$$i = 1, \ a_i(x) \equiv \alpha(x) \ ; \ a_i'(x) \equiv 0 \ ;$$

the functions Θ_{ai}, Θ_{bi} in (23) reduce to constants. Finally, the expressions

$$\frac{1}{2\pi i} \int_{(C_R)} d\rho \int_0^1 G'(x,t,\rho)f(t)dt \ \text{ and } \ S_N(f)$$

may differ but by a finite sum of terms of the form

$$\pm \ u_\nu^{(1)}(x) \int_0^1 f(t)v_\nu^{(1)}(t)dt$$

for which either $|\rho_\nu'| > R$ or $|\nu| > N$, and whose number does not exceed the number of the characteristic values (15) within $|\rho| = R_\delta$. It is readily seen that each of these terms $\to 0$ as $R \to \infty$, uniformly on $(0, 1)$.

35. THEOREM 9. *Under the conditions of Theorem 8 the resolvent kernel* $\Re(x, \xi, \rho)$ *of* (2) *admits of an expansion* (ρ *is not a characteristic value*)

$$\text{(24)} \qquad \Re(x,\xi,\rho) = \sum_{\nu=1}^{\infty} \Re_\nu(x,\xi,\rho)$$

where $\Re_\nu(x, \xi, \rho)$ *denotes the principal part of* $\Re(x, \xi, \rho)$ *corresponding to the pole* ρ_ν. *Accordingly, for the kernel* $K(x, \xi)$ *itself we have*

$$\text{(25)} \qquad K(x,\xi) = - \sum_{\nu=1}^{\infty} \Re_\nu(x,\xi,0),$$

the series of the left-hand member of (24) *and* (25) *being uniformly convergent in* (x, ξ) *on every closed part of the square* $(\mathfrak{S})$, *which does not contain any of the points* $(0, 1)$, $(1, 0)$ *and has no points in common with the line* $x = \xi$.

The solution of the non-homogeneous integral equation

$$\text{(26)} \qquad u(x) = f(x) + \rho \int_0^1 K(x,\xi)u(\xi)d\xi$$

admits of the expansion

$$\text{(27)} \qquad u(x) = f(x) - \rho \sum_{\nu=1}^{\infty} \int_0^1 \Re_\nu(x,\xi,\rho)f(\xi)d\xi$$

which is uniformly convergent on $(0, 1)$ *for any integrable* $f(x)$.

An easy application of the Cauchy fundamental theorem shows

$$\text{(28)} \qquad \Re(x,\xi,\rho) - G(x,\xi,\rho) - \sum_{(C_R)} \Re_\nu(x,\xi,\rho) + \sum_{(C_R)} G_\nu(x,\xi,\rho)$$

$$= \int_{(C_R)} \frac{\Gamma(x,\xi,\zeta) - G(x,\xi,\zeta)}{\zeta - \rho} d\zeta$$

where $G_\nu(x, \xi, \rho)$ denotes the principal part of $G(x, \xi, \rho)$ corresponding to the pole ρ_ν''. Since on (C_R)

$$\Gamma(x,\xi,\zeta) - G(x,\xi,\zeta) = O(1/\zeta)$$

(Theorem 3) the right-hand member of (28) $\to 0$ as $R \to \infty$, uniformly on $(0, 1)$. It is known from the general theory of the Green's function (D, 27) and may be proved directly in the present case that

$$G(x,\xi,\rho) = \sum_{(\nu)} G_\nu(x,\xi,\rho),$$

the series of the right-hand member being uniformly convergent on any region of the type mentioned in the statement of Theorem 9. Hence the same is true of the expansion (24). Expansion (25) follows from (24), for $\rho = 0$, since $K(x, \xi) = -\Re(x, \xi, 0)$.

Now, let $f(x)$ be any integrable function. For any fixed ρ which is not a characteristic value of (26), the solution of (26) is given by the formula

$$(29) \quad u(x) = f(x) - \rho \int_0^1 \Re(x,\xi,\rho)f(\xi)d\xi = f(x) - \rho \sum_{\nu=1}^\infty \int_0^1 \Re_\nu(x,\xi,\rho)f(\xi)d\xi,$$

the term by term integration being permissible by virtue of a known theorem of Lebesgue. To prove the uniform convergence of (29), take the difference

$$u(x) - f(x) + \rho \sum_{(C_R)} \int_0^1 \Re_\nu(x,\xi,\rho)f(\xi)d\xi$$

$$= -\frac{1}{2\pi i} \int_{(C_R)} \frac{d\zeta}{\zeta - \rho} \int_0^1 \Re(x,\xi,\zeta)f(\xi)d\xi$$

$$= -\frac{1}{2\pi i} \int_{(C_R)} \frac{d\zeta}{\zeta - \rho} \int_0^1 \{\Gamma(x,\xi,\zeta) - G(x,\xi,\zeta)\}f(\xi)d\xi$$

$$- \frac{1}{2\pi i} \int_{(C_R)} \frac{d\zeta}{\zeta - \rho} \int_0^1 G(x,\xi,\zeta)f(\xi)d\xi.$$

On account of Theorem 3 and Lemma 1 each term of the right side here is of the form

$$\int_{(C_R)} \frac{\epsilon d\zeta}{\zeta - \rho}$$

and hence $\to 0$, as $R \to \infty$, uniformly on $(0,1)$.

It should be noted that, in the case where all the poles of $\Re(x, \xi, \rho)$ are simple, formulas (24), (25), (27) reduce to well known expansions

$$\Re(x,\xi,\rho) = \sum_{\nu=1}^\infty \frac{u_\nu(x)v_\nu(\xi)}{\rho - \rho_\nu},$$

$$K(x,\xi) = \sum_{\nu=1}^\infty \frac{1}{\rho_\nu}u_\nu(x)v_\nu(\xi),$$

$$u(x) = f(x) - \rho \sum_{\nu=1}^\infty \frac{f_\nu u_\nu(x)}{\rho - \rho_\nu}, \quad f_\nu = \int_0^1 f(\xi)v_\nu(\xi)d\xi,$$

which, however, never before have been proved under our general conditions concerning the kernel $K(x, \xi)$.

We may leave to the reader the computation of the approximate expressions for the fundamental functions $u_\nu(x)$, $v_\nu(x)$ for ν large, as well as the applications of the results above to the integral equation (1).

36. To illustrate our general theory take the integral equation

$$(30) \qquad u(x) = \rho \int_0^1 K(x,\xi)u(\xi)d\xi \; ; \; K(x,\xi) = \begin{cases} mx^2\xi^2 + 2 \text{ if } \xi > x, \\ mx^2\xi^2 + 1 \text{ if } \xi < x, \end{cases}$$

which is analogous to that considered by Langer (L, pp. 638–639). An easy computation gives

$$\mathfrak{E}(x,\xi) = -\frac{4mx\xi^2}{2-m} \; ; \; a = -1 \; ; \; b = 2\frac{2+m}{2-m} \; ; \; \alpha(x) = \frac{8mx}{2-m} \; \cdot$$

Hence our theory can be applied to (30) unless $m = \pm 2$.

A direct computation shows that the characteristic values of (30) are the roots $(\neq 0)$ of the transcendental equation

$$e^\rho\{(6-3m)\rho^4 + 4m\rho^3 + 24m(\rho-1)^2\} = \rho^4(6m+12) + 4m(2\rho^3 - 3\rho^2 - 6\rho + 6)$$

and hence they are asymptotic to the roots of the equation

$$e^\rho = 2\frac{2+m}{2-m} = -\frac{b}{a} \qquad (m \neq \pm 2)$$

which agrees with Theorem 8.

The situation changes substantially if $m = \pm 2$. In this case the characteristic values of (30) are determined by the equations

$$e^\rho = \rho\left[\frac{3}{4}\right] \text{ if } m = 2,$$

$$e^\rho = \frac{1}{\rho}\left[-\frac{8}{3}\right] \text{ if } m = -2.$$

The asymptotic formula for ρ_ν involves logarithmic terms which are absent in the case where $m = \pm 2$. It should be noted that Langer's method also fails when $m = \pm 2$, since the asymptotic formula for ρ_ν as given by Langer does not contain logarithmic terms either.[*]

[*] It was not until recently that we noticed that a problem in some respects more general than that of Lichtenstein was treated by Mrs. Anna Pell-Wheeler in 1910. Cf. *Applications of biorthogonal systems of functions to the theory of integral equations*, these Transactions, vol. 12 (1911), pp. 165–180 (p. 176, Ex. 4). However, Mrs. Pell-Wheeler's problem does not include our problem (★) as a special case.

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A GENERAL THEORY OF NETS ON A SURFACE*

BY

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INTRODUCTION

In this paper we extend a method used by E. P. Lane in the study of conjugate nets† to the study of a non-conjugate, non-asymptotic net on a surface. We first refer the sustaining surface to the asymptotic net, and then refer it to an arbitrary net. We thus find that all of the projective properties of the net are expressible in terms of the quantities determining the character of the surface and two other functions. These two functions may be chosen arbitrarily, and hence they determine the most general net on the surface. It may be seen, from the form of the formulas involved, which properties of the net are really properties of the net, which are properties of the one-parameter families of curves forming the net, and which are properties entirely of the surface.

1. A SURFACE REFERRED TO ITS ASYMPTOTIC NET

Let

$$(1) \qquad y^{(k)} = y^{(k)}(u,v) \qquad\qquad (k = 1,2,3,4)$$

be the parametric equations of a surface S_y. If S_y is not a developable surface, and if the curves $u=$const., $v=$const. are the asymptotic curves, the four functions $y^{(k)}$ are solutions of two partial differential equations of the form

$$y_{uu} + 2ay_u + 2by_v + cy = 0,$$
$$y_{vv} + 2a'y_u + 2b'y_v + c'y = 0.$$

These two equations can be reduced to the form‡

$$(2) \qquad \begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0. \end{aligned}$$

The coefficients of system (2) satisfy the following integrability conditions:

$$a'_{uu} + g_u + 2a'_v b + 4a' b_v = 0,$$
$$b_{vv} + f_v + 2a' b_u + 4a'_u b = 0,$$
$$g_{uu} + 4b_v g + 2bg_v = f_{vv} + 4a'_u f + 2a' f_u.$$

The form (2) is not unique, but is preserved under all transformations of the form

$$\alpha = U(u), \quad \beta = V(v), \quad \bar{y} = C(U'V')^{1/2} y.$$

Any net on the surface may be defined by a differential equation of the form

(3) $$(\theta\, du - dv)(\omega\, dv - du) = 0, \quad 1 - \theta\omega \neq 0.$$

If $1 - \theta\omega$ were zero, the differential equation would represent only a one-parameter family of curves, and not a net. If $1 + \theta\omega = 0$, the net is a conjugate net. If $\theta = \omega = 0$ the net is the asymptotic net. We shall suppose, therefore, that

$$\theta\omega(1 - \theta^2\omega^2) \neq 0.$$

2. The surface referred to any non-conjugate net

Let us make the net (3) parametric by the transformation

(4) $$\alpha = \phi(u,v), \quad \beta = \psi(u,v),$$

for which

(5) $$\phi_u = -\theta\phi_v, \quad \psi_v = -\omega\psi_u.$$

If we make this transformation we obtain the system of differential equations of the form

(6) $$y_{\alpha\alpha} = \bar{a} y_{\alpha\beta} + \bar{b} y_\alpha + \bar{c} y_\beta + \bar{d} y,$$
$$y_{\beta\beta} = \bar{a}' y_{\alpha\beta} + \bar{b}' y_\alpha + \bar{c}' y_\beta + \bar{d}' y,$$

wherein

$$\bar{a} = \frac{2\omega}{1 + \theta\omega} \frac{\psi_u}{\phi_v}, \qquad \bar{a}' = \frac{2\theta}{1 + \theta\omega} \frac{\phi_v}{\psi_u},$$

$$\bar{b} = -\frac{\phi_{vv}}{\phi_v^2} + \frac{1}{\phi_v(1 - \theta^2\omega^2)} [2b\omega^2 + 2a'\theta + (\theta\theta_v - \theta_u)\omega^2],$$

$$\bar{c}' = -\frac{\psi_{uu}}{\psi_u^2} + \frac{1}{\psi_u(1 - \theta^2\omega^2)} [2a'\theta^2 + 2b\omega + (\omega\omega_u - \omega_v)\theta^2],$$

(7)
$$\bar{c} = - \frac{\psi_u}{\phi_v^2(1 - \theta^2\omega^2)}(2b\omega^3 + 2a' + \omega\omega_u - \omega_v),$$

$$\bar{b}' = - \frac{\phi_v}{\psi_u^2(1 - \theta^2\omega^2)}(2a'\theta^3 + 2b + \theta\theta_v - \theta_u),$$

$$\bar{d} = \frac{g - f\omega^2}{\psi_u^2(\theta^2\omega^2 - 1)}, \qquad \bar{d}' = \frac{f - g\theta^2}{\psi_u^2(\theta^2\omega^2 - 1)}.$$

System (6) defines the same surface as system (2), but the parametric net for (6) is the arbitrary non-conjugate net (3).

The following differentiation formulas will be found useful:

(8)
$$y_\alpha = \frac{\omega y_u + y_v}{(1 - \theta\omega)\phi_v}, \qquad y_\beta = \frac{y_u + \theta y_v}{(1 - \theta\omega)\psi_u}.$$

For convenience of reference we will give certain invariants and covariants of system (6). They are

(9)
$$\mathfrak{B} = \bar{a}'\bar{\gamma} - \tfrac{1}{2}(f_\beta - 2\bar{c}'), \quad \mathfrak{C} = \bar{a}\bar{\beta}' - \tfrac{1}{2}(f_\alpha - 2\bar{b})^*,$$

$$\rho = y_\alpha - \bar{\gamma}y, \quad \sigma = y_\beta - \bar{\beta}'y\dagger,$$

wherein

$$\bar{\gamma} = - \frac{\bar{c}}{\bar{a}}, \qquad \bar{\beta}' = - \frac{\bar{b}'}{\bar{a}'},$$

$$f_\alpha = \bar{b} + \frac{1}{1 - \bar{a}\bar{a}'}[\bar{a}_\beta + \bar{b} + \bar{a}'\bar{c} + \bar{a}(\bar{a}_\alpha' + \bar{a}\bar{b}' + \bar{c}')],$$

$$f_\beta = \bar{c}' + \frac{1}{1 - \bar{a}\bar{a}'}[\bar{a}_\alpha' + \bar{c}' + \bar{a}\bar{b}' + \bar{a}'(\bar{a}_\beta + \bar{a}'\bar{c} + \bar{b})].$$

The axis of the point y with respect to the net (3) joins y to the point ζ defined by$\ddagger$

(10)
$$\zeta = y_{\alpha\beta} - \bar{\beta}'y_\alpha - \bar{\gamma}y_\beta.$$

Substituting the values of the coefficients (7) into (9) and (10) we obtain these expressions in terms of θ, ω, and the coefficients and variables of system (2). They are

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* V. G. Grove, *A theory of a general net on a surface*, these Transactions, vol. 28 (1926), p. 496. Hereafter referred to as Grove, *Nets*.

† G. M. Green, *Nets of space curves*, these Transactions, vol. 21 (1920), p. 219. Hereafter referred to as Green, *Nets*.

‡ Green, *Nets*, p. 224.

$$\mathfrak{B} = \frac{1}{\omega(1-\theta\omega)^2(1+\theta\omega)\psi_u}\big[2(a'\theta + b\omega^2)(1-\theta^2\omega^2) - \omega(2+\theta\omega)(\theta\omega)_u$$
$$- (1+2\theta\omega)(\theta\omega)_v\big],$$

$$\mathfrak{C}' = \frac{1}{\theta(1-\theta\omega)^2(1+\theta\omega)\phi_v}\big[2(b\omega + a'\theta^2)(1-\theta^2\omega^2) - \theta(2+\theta\omega)(\theta\omega)_v$$
$$-(1+2\theta\omega)(\theta\omega)_u\big],$$

$$\rho = \frac{1}{(1-\theta\omega)\phi_v}\Big[\omega y_u + y_v - \frac{1}{2\omega}(2b\omega^3 + 2a' + \omega\omega_u - \omega_v)y\Big],$$

(11)

$$\sigma = \frac{1}{(1-\theta\omega)\psi_u}\Big[y_u + \theta y_v - \frac{1}{2\theta}(2a'\theta^3 + 2b + \theta\theta_v - \theta_u)y\Big],$$

$$\zeta = \frac{1+\theta\omega}{2\theta\omega(1-\theta\omega)^3\phi_v\psi_u}\big\{2\theta\omega(1-\theta\omega)y_{uv} - \big[2a'\theta(1-\theta^2\omega^2) + 2b\omega^2(1-\theta\omega)$$
$$- \omega(\theta\omega)_u - (\theta\omega)_v + \omega\theta_v(1-\theta\omega)\big]y_u - \big[2b\omega(1-\theta^2\omega^2) + 2a'\theta^2(1-\theta\omega)$$
$$- \theta(\theta\omega)_v - (\theta\omega)_u + \theta\omega_u(1-\theta\omega)\big]y_v + (\ )y\big\},$$

the coefficient of y being immaterial for our purposes.

3. The R-reciprocal congruences

Two congruences of importance in the theory of nets of curves are those congruences we have called the R-reciprocal congruences.* The lines of these congruences are reciprocal polars with respect to both of the quadrics osculating the parametric ruled surfaces of tangents at the surface point y. The lines of these congruences are also in relation R. The lines l and l' of the R-reciprocal congruences join the points

(12) $$r = y_\alpha - \frac{1}{2\bar a'}(f_\beta - 2\bar c')y, \qquad s = y_\beta - \frac{1}{2\bar a}(f_\alpha - 2\bar b)y,$$

and

(13) $$y, \quad z = y_{\alpha\beta} - \frac{1}{2\bar a}(f_\alpha - 2\bar b)y_\alpha - \frac{1}{2\bar a'}(f_\beta - 2\bar c')y_\beta.$$

The line l intersects the asymptotic tangents in the two points defined by

$$r_1 = y_u - \frac{1}{2\theta\omega(1-\theta\omega)}\big[2a'\theta^2(\theta\omega - 1) + \theta\omega(\theta\omega)_u + 2\theta(\theta\omega)_v$$
$$- \theta\omega_u(\theta\omega - 1)\big]y,$$

(14)

$$s_1 = y_v - \frac{1}{2\theta\omega(1-\theta\omega)}\big[2b\omega^2(\theta\omega - 1) + \theta\omega(\theta\omega)_v + 2\omega(\theta\omega)_u$$
$$- \omega\theta_v(\theta\omega - 1)\big]y.$$

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* Grove, *Nets*, p. 497.

The expression for z in terms of the coefficients and variables of system (2) is

$$z = \frac{4\theta\omega}{(1-\theta\omega)\phi_v\psi_u}\bigg\{ y_{uv} - \bigg[\frac{1}{2\theta\omega(1-\theta\omega)}(2b\omega^2(\theta\omega-1) + \theta\omega(\theta\omega)_v$$

$$+ 2\omega(\theta\omega)_u - \omega\theta_v(\theta\omega-1)) + \frac{(\theta\omega)_v}{1-\theta^2\omega^2}\bigg]y_u$$

(15)

$$- \bigg[\frac{1}{2\theta\omega(1-\theta\omega)}(2a'\theta^2(\theta\omega-1) + \theta\omega(\theta\omega)_u$$

$$+ 2\theta(\theta\omega)_v - \theta\omega_u(\theta\omega-1)) + \frac{(\theta\omega)_u}{1-\theta^2\omega^2}\bigg]y_v + (\ )y\bigg\},$$

the coefficient of y being immaterial for our purposes.

Now the two lines joining the points

$$r = y_u - \lambda y, \qquad s = y_v - \mu y,$$

and the point y to

$$z = y_{uv} - my_u - ny_v$$

are Green reciprocal lines* if, and only if, $m = \mu$ and $n = \lambda$. We note from (14) and (15) that the R-reciprocal congruences are also Green reciprocal congruences if, and only if, $\theta\omega = $ const.

From the first two of equations (7) we find

$$\bar{a}\bar{a}' = \frac{4\theta\omega}{(1+\theta\omega)^2}.$$

Hence *if $\theta\omega$ is constant then $\bar{a}\bar{a}'$ is constant, and conversely.* If $\bar{a}\bar{a}' = $ const. the tangents to the curves of the net form with the asymptotic tangents a constant cross ratio.† We may state our results in the following way: *The R-reciprocal congruences are Green reciprocal congruences if and only if the tangents to the curves of the net form with the asymptotic tangents a constant cross ratio.*

Let us now find the condition that the R-reciprocal congruences coincide with a given pair of Green reciprocal congruences. We must have $\theta\omega = $ const. Let the given lines join the points

* G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 86. Cited as Green, *Surfaces*.

† Grove, *Nets*, p. 500. This result was also derived by E. P. Lane in his paper *Bundles and pencils of nets on a surface*, these Transactions, vol. 28 (1926), p. 165.

$$r = y_u - \lambda y, \qquad s = y_v - \mu y,$$

and y to the point

$$z = y_{uv} - \mu y_u - \lambda y_v.$$

From (14) and (15) we have

$$\theta \omega_u - 2a'\theta^2 = 2\theta \omega \lambda, \quad \omega \theta_v - 2b\omega^2 = 2\theta \omega \mu.$$

Putting $\theta \omega = k$, where k is a constant, these may be written

$$(16) \qquad \frac{\theta_u}{\theta} = -\frac{2a'\theta^2}{k} - 2\lambda, \quad \frac{\theta_v}{\theta} = \frac{2bk}{\theta^2} + 2\mu.$$

A necessary and sufficient condition that (16) have a solution is that the equation

$$\frac{\partial}{\partial u}\left(\frac{bk}{\theta^2} + \mu\right) = \frac{\partial}{\partial v}\left(-\frac{a'\theta^2}{k} - \lambda\right)$$

be satisfied by such a solution. This condition of integrability may be written

$$(17) \qquad (a'_v + 4a'\mu)\theta^4 + k(\mu_u + \lambda_v + 8a'b)\theta^2 + k^2(b_u + 4b\lambda) = 0.$$

Two cases are possible:

    1. This equation may be an identity in θ, in which case the coefficients of the quadratic in θ^2 are all zero.

    2. The biquadratic may be solved for θ.

    In Case 1, we will assume that the surface is not ruled. Then

$$(18) \qquad \lambda = -\frac{b_u}{4b}, \quad \mu = -\frac{a'_v}{4a'}, \quad 8a'b + \mu_u + \lambda_v = 0.$$

Hence the assigned pair of Green reciprocal congruences is not arbitrary; the pair must be the canonical congruences as defined by Green.* The third equation of (18) shows that the surface itself is restricted. This equation may be written

$$(19) \qquad \frac{\partial^2}{\partial u \partial v} \log (a'b) = 32a'b.$$

If we put $a'b = e^\phi$, equation (19) may be written

$$(20) \qquad \frac{\partial^2 \phi}{\partial u \partial v} = 32e^\phi.$$

---

    * Green, *Surfaces*, p. 114.

The solution of (20) determines ϕ and hence the class of surfaces for which (19) holds. Since the solution of the completely integrable system (16) involves one arbitrary constant, we may state that *for each surface of the type defined by equation* (19) *the canonical congruences of the first and second kind will serve as the R-reciprocal congruences for a two-parameter family of nets on that surface.*

In Case 2, the surface may be arbitrary, the given pair of congruences being restricted. If we solve the biquadratic for θ and substitute the values of θ in equations (16), there result two differential equations of the second order which λ and μ must satisfy. These equations are of the form

$$
(21) \qquad
\begin{aligned}
\lambda_{uv} + \mu_{uu} &= G(\lambda,\mu,\lambda_u,\mu_u,\lambda_v,a',b), \\
\lambda_{vv} + \mu_{uv} &= G'(\lambda,\mu,\lambda_v,\mu_v,\mu_u,a',b),
\end{aligned}
$$

where G and G' are explicit functions of the indicated arguments. No restrictions are required for a solution of (21). Hence in Case 2, *there exists for any surface S_y whatever, a class of Green reciprocal congruences defined by* (21), *each pair of which will serve as R-reciprocal congruences for a net on a surface and in general for only one.*

4. Green reciprocal lines in relation R

Suppose we have a pair of lines in relation R with respect to the net (3). These lines may be obtained by joining the points $\bar{r}$ and $\bar{s}$ defined by

$$
(22) \qquad \bar{r} = y_\alpha - \lambda' y, \quad \bar{s} = y_\beta - \mu' y,
$$

and the point y to the point $\bar{z}$ defined by

$$
(23) \qquad \bar{z} = y_{\alpha\beta} - \mu' y_\alpha - \lambda' y_\beta.
$$

By methods similar to those used in §3, we find that if these lines are to be Green reciprocal lines we must have

$$
2a'\theta - \omega_u - \phi_v \frac{\partial(\theta\omega)}{\partial\alpha} - (\omega\mu'\psi_u + \lambda'\phi_v)(\theta\omega - 1) = (\lambda'\phi_v - \omega\mu'\psi_u)(\theta\omega + 1),
$$

$$
2b\omega - \theta_v - \psi_u \frac{\partial(\theta\omega)}{\partial\beta} - (\theta\lambda'\phi_v + \mu'\psi_u)(\theta\omega - 1) = (\mu'\psi_u - \theta\lambda'\phi_v)(\theta\omega + 1).
$$

Since these equations may be solved uniquely for λ' and μ' we may state that *associated with every non-conjugate, non-asymptotic net of curves on a non-developable surface there is one and only one pair of Green reciprocal congruences that are in relation R with respect to the net.*

5. The ray and axis of a point

The line joining the two points

$$\rho = y_\alpha - \bar{\gamma}y, \qquad \sigma = y_\beta - \bar{\beta}'y$$

has been called *the ray* of the point y with respect to the net[*]. The line joining y to the point

$$\zeta = y_{\alpha\beta} - \bar{\beta}'y_\alpha - \bar{\gamma}y_\beta$$

is called the *axis of the point* y with respect to the net.[†] Let us find the condition on the net in order that the ray (axis) coincide with an arbitrary line lying in the tangent plane at y (protruding from the surface at y).

Let the given arbitrary line lying in the tangent plane at y join the two points

(24) $$r = y_u - \lambda y, \qquad s = y_v - \mu y.$$

If we impose the condition that this line coincides with the line determined by ρ and σ defined by (11) we find it necessary and sufficient that θ and $1/\omega$ are solutions of

(25) $$\frac{\partial F}{\partial u} - F\frac{\partial F}{\partial v} = 2a'F^3 - 2\mu F^2 - 2\lambda F + 2b.$$

The net (3) will have the line (24) as ray.

Any line protruding from the surface at y joins y to the point

$$z = y_{uv} - \mu y_u - \lambda y_v.$$

If we impose the condition that this line coincide with the axis of y determined by y and Z of (11) we find it necessary and sufficient that θ and $1/\omega$ are any two solutions of the differential equation[‡]

(26) $$\frac{\partial F}{\partial u} + F\frac{\partial F}{\partial v} = -2a'F^3 + 2\mu F^2 - 2\lambda F + 2b.$$

Then the net (3) will have the arbitrary line yz as axis. In that case the curves C_α, C_β are union curves of the congruence of lines yz.

---

[*] Green, *Nets*, p. 232 and footnote.

[†] Ibid., p. 224 and footnote.

[‡] Equations similar to (25) and (26) were obtained by E. Bompiani in a paper *Sistemi coniugati e sistemi assiali di linee sopra una superficie dello spazio ordinario*, Bollettino della Unione Matematica Italiana, vol. 3 (1924). In this paper Bompiani considers the problem of determining among the system of union curves of a given congruence two one-parameter families of curves forming a conjugate net.

Let now the ray and axis of the point y with respect to the net (3) coincide with a pair of arbitrary Green reciprocal lines. Then equations (25) and (26) will hold simultaneously. If we add and subtract these equations in the proper order we obtain the two equivalent ones

$$(27) \qquad F_u = -2\lambda F + 2b, \quad F_v = 2\mu F - 2a'F^2.$$

A necessary and sufficient condition that equations (27) have a solution is that

$$\frac{\partial}{\partial v}(b - \lambda F) = \frac{\partial}{\partial u}(\mu F - a'F^2)$$

be satisfied by such a solution. We may write this equation in the form

$$(28) \qquad (a_u' - 2a'\lambda)F^2 + (4a'b - \lambda_v - \mu_u)F + (b_v - 2b\mu) = 0.$$

Two possibilities arise:

1. Equation (28) may be an identity in F.
2. The quadratic may be solved for F.

In Case 1 let us assume that the surface is not ruled; we have therefore

$$(29) \qquad \lambda = \frac{a_u'}{2a'}, \qquad \mu = \frac{b_v}{2b}, \qquad 4a'b - \lambda_v - \mu_u = 0.$$

The values for λ and μ show that the assigned pair of Green reciprocal congruences is not arbitrary; they must be the directrix congruences of the first and second kind. The third equation of condition shows that the surface is also restricted. This condition may be reduced to the form

$$(30) \qquad \frac{\partial^2}{\partial u \partial v} \log(a'b) = 8a'b.$$

If we put $\phi = \log(a'b)$, equation (30) may be written

$$(31) \qquad \frac{\partial^2 \phi}{\partial u \partial v} = 8e^\phi.$$

The solution of (31) determines the function ϕ and hence a class of surfaces for which (30) holds. For each solution ϕ, the function F may be obtained by a quadrature from (27). *Any two solutions $F_1(u, v, c_1)$, $F_2(u, v, c_2)$ determine a net such that the ray and axis of the point y with respect to the net are the directrices of the first and second kinds.*

In Case 2 we may solve the quadratic in F. Unless the roots are distinct, non-zero and finite, no net exists with the desired property. If we differentiate (28) with respect to u and v and use (27) we find that F must satisfy a second

quadratic in F and a cubic in F. Imposing the condition that these quadratics have the same roots and that these roots are two of the roots of the cubic, we find that λ and μ must satisfy equations of the form

$$
\begin{aligned}
\lambda_{vv} + \mu_{uv} &= G_1(\lambda_u, \lambda_v, \mu_u, \mu_v, \lambda, \mu, a', b), \\
\lambda_{uv} + \mu_{uu} &= G_2(\lambda_u, \lambda_v, \mu_u, \mu_v, \lambda, \mu, a', b), \\
0 &= G_3(\lambda_u, \lambda_v, \mu_u, \mu_v, \lambda, \mu, a', b), \\
0 &= G_4(\lambda_u, \lambda_v, \mu_u, \mu_v, \lambda, \mu, a', b).
\end{aligned}
$$

(32)

The integrability conditions of (32) restrict the surface S_y to be of a certain type. For surfaces of this type there exist Green reciprocal lines which will serve for the ray and axis of y with respect to a net determined by the roots of (28).

6. Pairs of conjugate nets

Let us now consider the one-parameter family of curves

(33) $$\theta\, du - dv = 0.$$

The direction conjugate to the direction defined by (33) is defined by

(34) $$\theta\, du + dv = 0.$$

Let us consider also the family of conjugate nets

(35) $$\theta^2 h^2\, du^2 - dv^2 = 0,$$

where h is an arbitrary constant but not zero. Wilczynski* has called such a one-parameter family of conjugate nets a pencil of conjugate nets. Wilczynski has also shown† that as this arbitrary net varies over all the nets of the pencil, the locus of the focal points of the tangents to the curves of the net at the point y is a cubic curve. The equation of this curve referred to the triangle y, y_u, y_v is

(36) $$C_\theta \equiv x_1 x_2 x_3 + b x_2^3 + a' x_3^3 - \frac{\theta_u}{2\theta} x_2^2 x_3 + \frac{\theta_v}{2\theta} x_2 x_3^2 = 0.$$

Similarly the locus of the focal points of the tangents to the curves of the pencil

(37) $$\omega^2 k^2\, dv^2 - du^2 = 0$$

* E. J. Wilczynski, *Geometrical significance of isothermal conjugacy of a net of curves*, American Journal of Mathematics, vol. 42 (1920), p. 217.

† E. J. Wilczynski, Oral Communication to the American Mathematical Society, December, 1920, cited as W(1920); Lane, *Nets*, p. 290.

is the cubic C_ω whose equation is

$$(38) \qquad C_\omega \equiv x_1 x_2 x_3 + b x_2^3 + a' x_3^3 + \frac{\omega_u}{2\omega} x_2^2 x_3 - \frac{\omega_v}{2\omega} x_2 x_3^2 = 0.$$

*The two curves C_θ and C_ω coincide if and only if $\theta\omega = $ const., that is, if and only if at every point of the surface the tangents to the net (3) form with the asymptotic tangents a constant cross ratio.**

If C_θ and C_ω are not identical, they intersect in nine points, the point y counting as eight points and one other point P. This ninth point lies on the line

$$(39) \qquad (\theta\omega)_u x_2 - (\theta\omega)_v x_3 = 0.$$

The nodal cubic C_θ has three inflection points. The line on which these points lie has been called the *flex-ray* of the point y with respect to the net (35). The equation of this line referred to the triangle y, y_u, y_v is

$$(40) \qquad x_1 - \frac{\theta_u}{2\theta} x_2 + \frac{\theta_v}{2\theta} x_3 = 0.$$

Similarly the flex ray of the point y with respect to the net (37) is the line whose equation is

$$(41) \qquad x_1 + \frac{\omega_u}{2\omega} x_2 - \frac{\omega_v}{2\omega} x_3 = 0.$$

If C_θ and C_ω do not coincide, the flex rays (40) and (41) intersect in a point which lies on the line

$$(39 \text{ bis}) \qquad (\theta\omega)_u x_2 - (\theta\omega)_v x_3 = 0.$$

Hence *the ninth point of intersection of the ray-point cubics of two conjugate nets, the intersection of the two flex rays and the point y are collinear.*

The envelope of the osculating planes of the curves of the pencil (35) has been called the axis cone.† Wilczynski has shown that this surface is a cone of class three and that the cone is the space dual of the nodal cubic. The equation of the axis cone with respect to the net (35) in plane coordinates referred to the tetrahedron y, y_u, y_v, y_{uv} is

$$K_\theta \equiv u_1 u_2 u_3 + a' u_2^3 + b u_3^3 - \frac{\theta_v}{2\theta} u_2^2 u_3 + \frac{\theta_u}{2\theta} u_2 u_3^2 = 0.$$

---

* If $\theta\omega = $ const. the nets $\theta^2 du^2 - dv^2 = 0$, $\omega^2 dv^2 - du^2 = 0$ belong to the same pencil of conjugate nets, and hence they determine the same ray-point cubic (Lane, *Nets*, p. 290).

† W (1920); Lane, *Nets*, p. 292.

Similarly the axis cone of y with respect to the net (37) has the equation

$$K_\omega \equiv u_1 u_2 u_3 + a' u_2^3 + b u_3^3 + \frac{\omega_v}{2\omega} u_2^2 u_3 - \frac{\omega_u}{2\omega} u_2 u_3^2 = 0.$$

If $\theta\omega$ is not constant, then K_θ and K_ω have nine common tangent planes, eight being accounted for at y. The ninth common tangent plane π passes through the point

(42) $(\theta\omega)_v u_2 - (\theta\omega)_u u_3 = 0.$

Each of the two cones K_θ and K_ω has three cusp axes which intersect in a line called by Wilczynski[*] the *cusp axis* of y. The plane of these two lines also passes through the point (42). Hence *the ninth common tangent plane of the axis cones of two conjugate nets, the plane of the cusp axes of the cones, and the tangent plane to S_v at y intersect in a line.* The equation of this line is

(43) $(\theta\omega)_u x_2 + (\theta\omega)_v x_3 = 0, \quad x_4 = 0.$

The two lines (39) and (43) are the tangents to the curves of a unique conjugate net whose differential equation is

(44) $(\theta\omega)_u^2 du^2 - (\theta\omega)_v^2 dv^2 = 0.$

The line (43) is the tangent to the curve

(45) $\phi(u,v) = \theta\omega = \text{const}.$

The family of curves defined by (45) is such that the cross ratio of the tangents to the curves of the net (3) and the asymptotic tangents is the same constant for points along the individual curves of the family.

We may associate a third cubic curve with the net (3) by means of the associate conjugate net[†] of the net (3). The differential equation of this net is readily seen to be

(46) $\theta \, du^2 - \omega \, dv^2 = 0.$

The ray-point cubic of the point y with respect to the net (46) has the equation

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[*] W (1920); Lane, *Nets*, p. 292.

[†] The associate conjugate net of a net has been defined by Green for any net on a surface as the net the tangents to whose curves are the double rays of the involution determined by the tangents to the curves of the given net and the asymptotic tangents.

$$C_{\theta\omega} \equiv x_1x_2x_3 + bx_2^3 + a'x_3^3 - \frac{1}{4}\left(\frac{\theta_u}{\theta} - \frac{\omega_u}{\omega}\right)x_2^2 x_3$$

$$+ \frac{1}{4}\left(\frac{\theta_v}{\theta} - \frac{\omega_u}{\omega}\right)x_2 x_3^2 = 0.$$

The flex ray of the point y with respect to $C_{\theta\omega}$ has the equation

(47) $$x_1 - \frac{1}{4}\left(\frac{\theta_u}{\theta} - \frac{\omega_u}{\omega}\right)x_2 + \frac{1}{4}\left(\frac{\theta_v}{\theta} - \frac{\omega_v}{\omega}\right)x_3 = 0.$$

The cubic $C_{\theta\omega}$ is one of the curves of the pencil determined by C_θ and C_ω and the line (47) is a line of the pencil determined by the flex rays (40) and (41). Hence *the ninth point of intersection of the cubics C_θ, C_ω, $C_{\theta\omega}$ and the intersections of the three flex rays are collinear with y.* As a matter of fact any curve of the pencil determined by C_θ and C_ω has a flex ray which is one of the lines of the pencil determined by the flex rays of C_θ and C_ω. The Hessians of the curves of the pencil determined by C_θ and C_ω have their common ninth point on the line (41). The corresponding dual theorems are also true.

7. A ONE-PARAMETER FAMILY OF CURVES

Let us now consider the case in which the given net is composed of an arbitrary one-parameter family of curves and one of the families of asymptotic curves. Let the given net be

(48) $$(\theta\,du - dv)dv = 0.$$

We may make the net (48) parametric by the transformation

(49) $$\alpha = \phi(u,v), \quad \beta = v, \quad \phi_u = -\theta\phi_v.$$

From (49) we derive the differentiation formulas

(50) $$y_\alpha = -\frac{y_u}{\theta\phi_v}, \qquad y_\beta = \frac{y_u + \theta y_v}{\theta},$$

from which we obtain

(51) $$y_{\alpha\beta} = -\frac{1}{\theta^3\phi_v}(\theta^2 y_{uv} - \theta_u y_u - 2b\theta y_v - f\theta y).$$

Consider the line l joining the points $\bar{r}$ and $\bar{s}$ defined by

(52) $$\bar{r} = y_\alpha - \bar{\lambda}y, \qquad \bar{s} = y_\beta - \bar{\mu}y.$$

The line l' in relation R to l with respect to the net (48) joins y to the point[*]

(53) $$\bar{z} = y_{\alpha\beta} - \bar{\mu}y_\alpha - \bar{\lambda}y_\beta.$$

If we use (50) and (51), (52) and (53) may be written

$$- \theta\phi_v\bar{r} = y_u - \lambda y, \quad \theta\bar{s} = y_u + \theta y_v - \bar{\mu}\theta y,$$

(54) $$- \theta\phi_v\bar{z} = y_{uv} - \frac{1}{\theta}\left(\frac{\theta_u}{\theta} + \theta\bar{\mu} + \lambda\right)y_u - \left(\frac{2b}{\theta} + \lambda\right)y_v,$$

wherein

$$\lambda = -\theta\phi_v\lambda'.$$

The line l intersects the asymptotic tangents in the points

$$r = y_u - \lambda y, \qquad s = y_v - \frac{1}{\theta}(\theta\mu' - \lambda)y.$$

The lines l and l' will therefore be Green reciprocal lines if, and only if, $b = 0$ and $\lambda = -\theta_u/(2\theta)$. Hence *to each curve of a one-parameter family through y on a ruled surface S_y there corresponds one and only one point r on S_y through which lines may be drawn whose Green reciprocal lines are in relation R with respect to the net composed of the given one-parameter family and the generators of S_y. This point lies on the generator through y. Moreover, as may be shown, the lines of this pencil have coplanar Green reciprocal lines.*

This theorem suggests that certain interesting results might be obtained if the point y were a flecnode or complex point, and the given curve through y the flecnode or complex curve. These investigations are beyond the scope of this paper and we will leave them for some future time.

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* Grove, *Nets*, p. 493.

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SIMPLY TRANSITIVE PRIMITIVE GROUPS[*]

BY

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1. The transitive constituents of the subgroup that leaves fixed one letter of a transitive group occur in pairs of equal degree. Transitive constituents on one letter are to be taken into account in the above statement. The two members of a pair sometimes coincide. This important property of transitive groups was proved by means of a certain quadratic invariant by Burnside in 1900.[†] It can be more easily demonstrated as follows:

Let G be a non-regular transitive group and let the subgroup G_1 of order g_1 that fixes one letter a of G have the transitive constituents B on the letters $b, b_1, \cdots, b_{r-1}$; C on the letters $c, c_1, \cdots, c_{s-1}$; $\cdots$. Consider a permutation $S = (cab \cdots) \cdots$. Every one of the g_1 permutations G_1S replaces a by b; and because $c, c_1, \cdots$ are the letters of a transitive constituent of G_1, every permutation G_1S is of the form $(c'ab \cdots) \cdots$, where c' is some one of the letters $c, c_1, \cdots$. Similarly every permutation SG_1 is of the form $(cab' \cdots) \cdots$. Then the array G_1SG_1 includes every permutation of G in which a is preceded by one of the s c''s or is folowed by one of the r b''s. Now the number of distinct permutations in the array G_1SG_1 is g_1^2 divided by the number of permutations common to G_1 and SG_1S^{-1}, or common to $S^{-1}G_1S$ and G_1,[‡] that is, by the number of permutations of G_1 that fix c or b. These numbers are g_1/s and g_1/r. Therefore $r = s$. If, as often happens, every permutation $(ab \cdots) \cdots$ is of the form $(ab) \cdots$ or $(b_1ab \cdots) \cdots$, the transitive constituent B is paired with itself. Since the product

$$(a)(bb_1 \cdots) \cdots (b_1ab \cdots) \cdots = (ab) \cdots ,$$

G, whenever a transitive constituent of G_1 is paired with itself, is of even order. A properly chosen odd power of this product is a permutation $(ab) \cdots$ of order a power of 2.

2. As in §1, G is a transitive group of order g and G_1 is a subgroup that fixes one of the n letters of G. Let H be a subgroup of G_1 of degree $n-m$ $(0 < m < n)$ and let I be the largest subgroup of G in which H is invariant.[§] If H is one of r conjugate subgroups in G_1, the largest subgroup of G_1 in which

H is invariant is of order g/nr and the order of I is gm/ns, where s is the total number of conjugates of H under G found in G_1. Then I has a transitive constituent of degree mr/s in letters fixed by H. The letters of this constituent are a, a_1, $\cdots$ and a is the letter fixed by G_1. Every permutation $(aa_1 \cdots)$ $\cdots$ of G transforms H into one of its r conjugates H, H', $\cdots$ under G_1. A second set of r_1 conjugate subgroups of G_1 is H_1, H_1', $\cdots$. Each of the g_1 permutations $(ab \cdots) \cdots$ of G transforms one of the r_1 subgroups H_1, H_1', $\cdots$ into H, so that in G_2, the subgroup of G that fixes b, H is one of r_1 conjugates and I has a transitive constituent of degree mr_1/s on the letters b, b_1, $\cdots$. The two sets of letters a, a_1, $\cdots$ and b, b_1, $\cdots$ do not coincide. If there is a third conjugate set of r_2 members H_2, H_2', $\cdots$ in G_1, there is a third transitive constituent c, c_1, $\cdots$ of degree mr_2/s in I, and so on. It is this correspondence between the conjugate sets H, H', $\cdots$; H_1, H_1', $\cdots$; $\cdots$ of G_1 and the transitive constituents a, a_1, $\cdots$; b, b_1, $\cdots$; $\cdots$ of I in the letters fixed by H, which is to be borne in mind in the developments of the following sections.

3. Examples of transitive groups in illustration of the preceding theory may be helpful.

$$G_{10}^{5} = \left\{ (bb_1)(cc_1),\ (ab)(b_1c) \right\}.$$
$$G_{21}^{7} = \left\{ (bb_1b_2)(cc_1c_2),\ (abc)(b_1c_2b_2) \right\}.$$
$$G_{72}^{9} = \left\{ (bb_1)(b_2b_3)(cc_1),\ (bb_2)(cc_2)(c_1c_3),\ (ab)(b_1c_2)(b_3c_1) \right\}.$$
$$G_{120}^{10} = \left\{ (bb_1)(cc_4)(c_1c_5),\ (bb_2)(c_2c_4)(c_3c_5),\ (cc_1)(c_2c_3)(c_4c_5), \right.$$
$$\left. (ab)(b_1c_5)(b_2c_4)(c_1c_2) \right\}.$$

These four groups are primitive. In the one of degree 10 the subgroup that leaves a fixed is the following:

$1,$	$(cc_1)(c_2c_3)(c_4c_5),$
$(bb_1b_2)(cc_3c_4c_1c_2c_5),$	$(bb_1b_2)(cc_2c_4)(c_1c_3c_5),$
$(bb_2b_1)(cc_5c_2c_1c_4c_3),$	$(bb_2b_1)(cc_4c_2)(c_1c_5c_3),$
$(bb_1)(cc_4)(c_1c_5),$	$(bb_1)(cc_5)(c_1c_4)(c_2c_3),$
$(bb_2)(c_2c_4)(c_3c_5),$	$(bb_2)(cc_1)(c_2c_5)(c_3c_4),$
$(b_1b_2)(cc_2)(c_1c_3),$	$(b_1b_2)(cc_3)(c_1c_2)(c_4c_5).$

4. *In all that follows G is a simply transitive primitive group.* G_1 is intransitive of degree $n-1$ and is a maximal subgroup of G. Let H be invariant in G_1 and of degree $< n-1$. Then I and G_1 coincide and I may be said to have one transitive constituent on the one letter a fixed by G_1. Hence the factor m/s of §2 is unity. Since H is invariant in G_1, it must fix all the letters of

one or more transitive constituents of G_1. Because G_1 does not fix two letters of G, of all the conjugates of H under G, only H is invariant in G_1. Therefore H, like the letter a, is characteristic of G_1. It is not, however, to be inferred from this statement that H is a "characteristic" subgroup of G_1 in the strong sense of being invariant in the holomorph of G_1. What is meant is that the n conjugate subgroups H, $\cdots$ are in one-to-one correspondence to the n subgroups G_1, $\cdots$ and therefore are in one-to-one correspondence to the n letters a, $\cdots$ of G. In G_1 there are exactly $m-1$ non-invariant subgroups H_1, $\cdots$, and "they are transformed by G_1 in the same manner as the letters of one of G_1's constituent groups of degree $m-1$."* The constituent group may be transitive or intransitive. This well known conclusion leaves open the question as to whether or not this constituent of G_1 according to which the $m-1$ subgroups H_1, $\cdots$ are permuted contains letters displaced by H.† This is an unsolved problem of fundamental importance.

If the transitive constituent B (on letters fixed by H) is paired with itself in the sense of §1, the permutation $S=(ab)\ \cdots$, known to exist in G, which transforms H_1 into H, has an inverse $S^{-1}=(ab)\ \cdots$ which transforms H into H_1 and which transforms some member H_1' of the conjugate set H_1, H_1', $\cdots$ of G_1 into H (§2). Now H_1 is the invariant subgroup of the subgroup G_2 that fixes b and in which H is included. The r_1 conjugate subgroups H_1, H_1', $\cdots$ are therefore permuted according to the permutations of the transitive constituent B of G_1. Conversely, if in the permutation $S=(ab)\ \cdots$ of §2 which transforms H_1 into H, the letter b is one of the transitive set according to which H_1, H_1', $\cdots$ are permuted by G_1, B is paired with itself.

If two transitive constituents B and C, both in letters not displaced by H, are paired, there is in G a permutation $S=(cab\ \cdots)\ \cdots$ such that

$$S^{-1}H_1S = H, \qquad SH_2S^{-1} = H \ ;$$

and hence

$$SHS^{-1} = H_1, \qquad S^{-1}HS = H_2.$$

Then H_1, H_1', $\cdots$ are permuted according to the transitive constituent C, and H_2, H_2', $\cdots$ are permuted according to the transitive constituent B. Here also the converse is true.

5. THEOREM I. *If all the transitive constituents of H are of the same degree, or if no two (not of maximum degree in H) belong to the same transitive constituent of G_1, every subgroup of G_1 similar to H is transformed into itself by H.*

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* Miller, Proceedings of the London Mathematical Society, vol. 28 (1897), p. 535.

† Rietz, American Journal of Mathematics, vol. 28 (1904), p. 10, line 23.

The above conclusion is equivalent to the statement that the constituent of G_1 according to which the $m-1$ subgroups H_1, H_1', $\cdots$ are permuted displaces no letter of H. For if H_1, H_1', $\cdots$ are permuted according to the constituent B of G_1 on the r_1 letters b, b_1, $\cdots$ fixed by H, the subgroup of G_1 that fixes b, say, is the largest subgroup of G_1 in which H_1 is invariant and includes H. Conversely, if each of the r_1 subgroups H_1, H_1', $\cdots$ is transformed into itself by every permutation of H, the transitive constituent of degree r_1 of G_1 according to which they are permuted displaces no letter of H.

The letter of G fixed by G_1 is a. Let b, b_1, $\cdots$ be certain letters fixed by H but permuted transitively by G_1. Let α, α_1, $\cdots$, β, β_1, $\cdots$, $\cdots$ be the letters of some transitive constituent of G_1 and such that α, α_1, $\cdots$ is one transitive constituent of H, β, β_1, $\cdots$ is another, and so on. Of course if H displaces one letter of a transitive constituent of G_1 it displaces every letter of that constituent.

Now transform G_1 into G_2 by means of a permutation $(ab \cdots) \cdots$ of the primitive group G. At the same time H, an invariant subgroup of G_1, is transformed into an invariant subgroup of G_2. Call the latter subgroup H_b. We wish to show that H_b is necessarily a subgroup of G_1. Suppose it is not a subgroup of G_1. Then since a primitive group is generated by a subgroup leaving one letter fixed and any permutation of the group not in that subgroup, $\{G_1, H_b\} = G$. But if H_b fails to connect transitively letters of H and letters fixed by H, $\{G_1, H_b\}$ is intransitive. Hence, if H_b is not a subgroup of G_1, at least one of its permutations unites letters of H and letters fixed by H. Let us now impose upon the transitive constituent α, α_1, $\cdots$ of H the condition that no transitive constituent of H is of higher degree. The set β, β_1, $\cdots$, being in the same transitive constituent of G_1, will have exactly the same number of letters as the set α, α_1, $\cdots$. If α and x (let x be one of the m letters a, b, b_1, $\cdots$ fixed by H) are in the same transitive constituent of H_b, so also are all the other letters α_1, α_2, $\cdots$ of that transitive constituent of H. For since H fixes b, it is a subgroup of G_2, and in consequence every permutation of H transforms H_b into itself. Then H_b has a transitive constituent x, α, α_1, $\cdots$ of higher degree than any transitive constituent of H, to which H_b is conjugate under G; —an absurd result. Similarly the constituent β, β_1, $\cdots$ of H_b displaces no letter fixed by H. If H_b fixes all the letters α, β, $\cdots$ of one transitive constituent of G_1, the group $\{G_1, H_b\}$ is intransitive. Then H_b connects the letters α, β, $\cdots$ only with letters of H. Thus if all the transitive constituents of H are of the same degree the theorem is proved.

The letters λ, λ_1, $\cdots$ of a transitive constituent of H of lower degree are by hypothesis the letters of a transitive constituent of G_1. The transitive

constituent x, λ, λ_1, $\cdots$ of H_b contains all the letters of the transitive constituent λ, λ_1, $\cdots$ of G_1. Then the transitive constituents α, $\cdots$ and x, λ, $\cdots$ of H_b are not united by G_1.

COROLLARY I. *If G_1 has only two transitive constituents and contains an invariant subgroup H of degree $<n-1$, every subgroup of G_1 similar to H is transformed into itself by H, and G is of even order.*

In this case H is an invariant intransitive subgroup of an imprimitive group and all its transitive constituents are of the same degree. It is of even order because each transitive constituent of G_1 is paired with itself.

It was proved by Rietz[*] that if G is of odd order and if G_1 has only two transitive constituents, G_1 is a simple isomorphism between its two constituents.

COROLLARY II. *If G is of even order and G_1 has only two transitive constituents, each transitive constituent of G_1 is paired with itself.*

For G certainly contains a permutation $(ab)\ \cdots$ of order 2 which pairs one of the transitive constituents (B) with itself.

COROLLARY III. *If G_1 has three and only three transitive constituents and contains an invariant subgroup H of degree $<n-1$, every subgroup of G_1, similar to H, is transformed into itself by H.*

This is true except perhaps when H has transitive constituents α, α_1, $\cdots$, β, β_1, $\cdots$, $\cdots$ of degree t; and transitive constituents λ, λ_1, $\cdots$, μ, μ_1, $\cdots$, $\cdots$ of degree v ($<t$). The letters fixed by H are b, b_1, $\cdots$ of the transitive constituent B.

Assume as before that H_b displaces a, that is, H_b is not a subgroup of G_1. If the only letters in the transitive constituent of H_b with a are letters of B, G contains a permutation $(b'ab_1 \cdots) \cdots$, where b' is a letter of B, and the constituent B is paired with itself. This would prove the Corollary, so that H_b has a transitive constituent a, λ, $\cdots$ of degree t. Then G_1 is transformed into G_2 by a permutation $S = (\alpha ab \cdots) \cdots$ which pairs the constituents b, b_1, $\cdots$ and α, α_1, $\cdots$ of G_1. If the transitive constituent a, λ, $\cdots$ of H_b contains a letter of B, H_b contains a permutation $(ab_1 \cdots) \cdots$ in which a is preceded by a letter of B or by one of the letters λ, $\cdots$, μ, $\cdots$. But every permutation of G which replaces a by a b belongs to the array G_1SG_1, in which only α's precede a. Now the transitive constituent a, λ, λ_1, $\cdots$ of H_b is of degree $t = kv+1$ if it contains letters of k transitive constituents of H. In one of the transitive constituents of H_b are found letters α and letters μ,

* Rietz, loc. cit., p. 11, Theorem 10.

for only thus can H_b unite these two transitive constituents of G_1. This transitive constituent of degree v in H_b cannot displace all the t letters $\alpha, \alpha_1, \cdots, \alpha_{t-1}$. But H_b, being transformed into itself by H, displaces all the $t+v$ letters $\alpha, \alpha_1, \cdots, \mu, \mu_1, \cdots$. H permutes q (say) transitive constituents $\alpha, \mu, \cdots$ of H_b. This means that the transitive constituent α of H has q systems of imprimitivity of t/q letters and that the transitive constituent μ of H has q systems of imprimitivity of v/q letters. That t and v should have a common factor q is inconsistent with the preceding result, $t = kv + 1$. Therefore H_b is a subgroup of G_1.

6. THEOREM II. *Let G be a simply transitive primitive group in which each of the m subgroups $H, H_1, \cdots$ of G_1 is transformed into itself by every permutation of H, the invariant subgroup of G_1. If the degree of the group generated by the complete set of conjugates $H_1, H_1', \cdots$ of G_1 is less than $n-1$, the letters of G_1 left fixed by it are the letters of one or more of the transitive constituents whose letters are already fixed by H.*

The group K generated by the r_1 conjugates $H_1, H_1', \cdots$ is an invariant subgroup of G_1. By hypothesis K is of degree $< n - 1$. There is therefore in G_1 a complete set of conjugate subgroups $K_1, K_1', \cdots$, similar to K, permuted according to a transitive constituent X of G_1. Now K_1 is conjugate to K under transformation by some permutation of G, so that K_1 is generated by some of the n subgroups $H, H_1, \cdots$. Because K_1 fixes a, its r_1 generating subgroups are subgroups of G_1. All the permutations of X except the identity actually permute two or more of the subgroups $K_1, K_1', \cdots$, conjugate under G_1. Then the identity is the only permutation of X that can occur in a constituent of H, because by hypothesis every permutation of H transforms each of the m subgroups $H, H_1, \cdots$ of G_1 into itself. Then X is one of the transitive constituents $B, C, \cdots$ of G_1 that displace no letter of H. What is true of $K_1, K_1', \cdots$ is true of all such conjugate sets of non-invariant subgroups of G_1 similar to K. Hence the only letters of G_1 fixed by K are letters already fixed by H.

7. THEOREM III. *If only one transitive constituent of G_1 is an imprimitive group (of order f), G_1 is of order f.*

Let B be the imprimitive constituent of G_1. Suppose that a subgroup H of G_1 corresponds to the identity of B. All the $n - m$ letters of v primitive constituents of G_1 are displaced by H. The $m - 1$ other letters of G_1 are distributed among w transitive constituents $B, C, \cdots$. Since an invariant subgroup of a primitive group is transitive, no two transitive constituents of H belong to the same transitive constituent of G_1. Then by Theorem I, each of the $m - 1$ non-invariant subgroups $H_1, H_1', \cdots$ similar to H of G_1 is

transformed into itself by H. By Theorem II, the group K generated by the conjugate set H_1, H_1', $\cdots$ of G_1 displaces all the letters of H. Since K is not H or a subgroup of H, K also displaces the letters of the imprimitive constituent B. If one of the generators H_1, H_1', $\cdots$ of K fixes all the letters of a transitive constituent of G_1, K fixes all the letters of that constituent. Hence H_1 has $v+1$ or more transitive constituents. Under G, H is conjugate to H_1 and therefore also has $v+1$ or more transitive constituents. But H displaces the letters of v primitive constituents of G_1 and has exactly v transitive constituents. Hence it is impossible that the order of G_1 exceeds that of the imprimitive constituent B.

COROLLARY I. *If all the transitive constituents of G_1 are primitive groups, G_1 is a simple isomorphism between its transitive constituents.*

Each of the primitive constituents of G_1 may be put in turn in the place of the imprimitive constituent of Theorem III.

COROLLARY II. *If G_1 has an intransitive constituent of order f, and if all the other transitive constituents of G_1 are primitive groups, G_1 is of order f.*

8. It has been known since 1921 that if one of the transitive constituents of G_1 of maximum degree is doubly transitive, G_1 is a simple isomorphism between its transitive constituents. Moreover the transitive constituents are similar groups whose corresponding permutations are multiplied together. For this is an immediate consequence of the following theorem:*

THEOREM IV. *Let G_1 have a t-ply $(t \geq 2)$ transitive constituent of degree m. If G_1 has no transitive constituent whose degree is a divisor $(>m)$ of $m(m-1)$, all the transitive constituents of G_1 are similar groups of order g/n.*

To guard against misunderstanding, we recall that two groups G and G' are *equivalent* when there exists a permutation by which one can be transformed into the other; and that two equivalent groups are *similar* when, if S_i and S_i' are corresponding permutations in some isomorphism of G to G', a permutation T exists such that $T^{-1}S_iT=S_i'$ $(i=1, 2, \cdots, g)$.† In the group G_{72}^9 of §3, the two octic constituents of G_1 are isomorphic and equivalent but are not similar. In G_{10}^5 and G_{21}^7 of §3, the two constituents of G_1 are similar.

A useful set of theorems having to do with simply transitive primitive groups was given by Dr. E. R. Bennett in 1912.‡ In particular, Corollary II to Theorem V, page 6, reads:

* Manning, *Primitive Groups*, 1921, p. 83.
† Manning, *Primitive Groups*, 1921, p. 39.
‡ E. R. Bennett, American Journal of Mathematics, vol. 34 (1912), p. 1.

"*If the transitive constituent of degree m in G_1 is a t-times transitive group* ($t \geqq 3$), G_1 *always contains an imprimitive group of degree* $m(m-1)$."

This result, in common with Dr. Bennett's Theorems I to VI, is subject to the following conditions upon the simply transitive primitive group G (of degree n) and its maximal subgroup G_1:

(1) The constituent M (of degree m) of G_1 is a non-regular transitive group.

(2) M is the only transitive constituent of G_1 whose degree divides m.

(3) Corresponding to the identity of M there is a subgroup H in G_1 of degree $n-m-1$.

This corollary raises interesting questions as to possible extensions of our Theorem IV, the proof of which is based merely upon the hypothesis that one constituent of G_1 is (at least) doubly transitive. Conditions (2) and (3) may be replaced by the weaker conditions of the following theorem:

THEOREM V. *Let G_1, the subgroup that leaves fixed one letter of the simply transitive primitive group G of degree n and order g, have a primitive constituent M of degree m, in which the subgroup M_1 that fixes one letter is primitive. Let M be paired with itself in G_1 and let the order of M be $<g/n$. Then G_1 contains an imprimitive constituent in which there is an invariant intransitive subgroup with m transitive constituents of $m-1$ letters each, permuted according to the permutations of the primitive group M.*

The letter of G fixed by G_1 is x, and the letters of M are $a, a_1, \cdots, a_{m-1}$. The subgroup of G that fixes both x and a is F. In F, $a_1, a_2, \cdots, a_{m-1}$ are the letters of a primitive constituent group. Because M is paired with itself in G_1 (§1), G contains a permutation $S = (xa) \cdots$ which transforms F into itself, G_1 into G_2 (fixing a), M of G_1 into a transitive constituent of G_2 on the letters $x, b_1, \cdots, b_{m-1}$, and the transitive constituent $a_1, a_2, \cdots, a_{m-1}$ of F into a transitive constituent $b_1, b_2, \cdots, b_{m-1}$ of F. These two transitive constituents of F are distinct because if they have one letter in common they have every letter in common and therefore $\{G_1, G_2\}$ would permute the $m+1$ letters $x, a, a_1, \cdots, a_{m-1}$ only among themselves, making G either intransitive or doubly transitive, contrary to hypothesis. Nor can the letters $b_1, b_2, \cdots, b_{m-1}$ be the letters of a transitive constituent of G_1. For if so, $\{G_1, G_2\}$ has a transitive constituent of degree m in the letters $x, b_1, \cdots, b_{m-1}$. The letters $b_1, b_2, \cdots$ belong to a transitive constituent (P) of G_1 of degree $\geqq m$.

There is an invariant subgroup H in G_1 corresponding to the identity of M. Because M is paired with itself in G_1, there is a complete set of m conjugate subgroups $H_1, H_1', \cdots$, similar to H, in G_1 which are permuted

according to the transitive constituent M (§4). Let H_1 correspond to the letter a of M. The largest subgroup of G_1 in which H_1 is invariant is F; H and H_1 are the invariant subgroups of G_1 and G_2 respectively. Since H_1 is a subgroup of G_1 and is not a subgroup of H, it displaces at least one letter of M, and since it is invariant in F it displaces the $m-1$ letters $a_1, a_2, \cdots, a_{m-1}$. Since $S^{-1}H_1S = H$, H displaces the $m-1$ letters $b_1, b_2, \cdots, b_{m-1}$. Now P (of degree $\geq m$) has an invariant intransitive subgroup in H with one transitive constituent of degree $m-1$. It is therefore an imprimitive group with systems of $m-1$ letters each. The only permutations of G_1 that permute these $m-1$ letters $b_1, b_2, \cdots, b_{m-1}$ among themselves are the permutations of F, the subgroup of G_1 that fixes a. Because M is primitive, F is a maximal subgroup of G_1 and is one of m conjugates under G_1. Then there are m systems of $m-1$ letters each in P and they are permuted according to a primitive group of degree m. That this primitive group is exactly M is evident from a consideration of the m conjugate subgroups $F, F_1, \cdots$. For F fixes a and fixes the constituent $b_1, b_2, \cdots, b_{m-1}$ of P, F_1 fixes a_1 and the constituent $b_1', b_2', \cdots, b_{m-1}'$ of P, and so on.

9. It is worth while to extend Dr. Bennett's Theorems I to VI, replacing the three given conditions by the single condition that M *is a transitive constituent of G_1 "paired with itself,"* and adding other limitations only as needed. We shall use the notation of the preceding section (§8).

Suppose G_1 has a transitive constituent Q of degree q. This constituent Q is transformed by S into a transitive constituent of G_2 which must include at least one letter new to Q.

If F permutes the letters of Q transitively, S transforms F into itself and therefore transforms this transitive constituent (A) of F on the letters of Q into a second transitive constituent (B) of F. There is no letter of Q in B. Since G_1 and G_2 cannot have transitive constituents on the same letters, G_1 has a transitive constituent of degree $>q$ in which all the letters of B occur.

If F does not permute the letters of Q transitively, the letters of at least one transitive constituent of degree l (≥ 1) of a subgroup of Q found in F is replaced by S by letters new to Q.

Instead of Q, consider now M and its subgroup M_1 that fixes a. The permutation S transforms the constituent M of G_1 into a transitive constituent of G_2 on the letters $x, b_1, b_2, \cdots, b_{m-1}$. The letters $b_1, b_2, \cdots, b_{m-1}$ do not coincide with the letters $a_1, a_2, \cdots, a_{m-1}$, for $\{G_1, G_2\}$ is simply transitive. If M_1 is transitive, $\{F, S\}$ has an imprimitive constituent with the two distinct systems $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$. If M_1 is intransitive,

or if M is regular, at least one transitive constituent $b_1, b_2, \cdots, b_l$ of $S^{-1}M_1S$ ($l \geq 1$) contains none of the letters $a_1, a_2, \cdots, a_{m-1}$.

We now impose the condition that *the order of M is $<g/n$.*

The invariant subgroup H of G_1, corresponding to the identity of M, is conjugate under G to m subgroups $H_1, H_1', \cdots$ which G_1 permutes according to the transitive constituent M. The subgroup H_1, say, is an invariant subgroup of G_2, and must displace some $j(>1)$ letters of M: $a_1, a_2, \cdots, a_j$. Thus if M is a regular group, its order is g/n.[*] The transform of H_1 by S is H. Let $b_1, b_2, \cdots, b_j$ be the letters by which S replaces $a_1, a_2, \cdots, a_j$. By definition H fixes all the letters of M. All the letters of the transitive constituents of G_1 to which $b_1, b_2, \cdots, b_j$ belong are displaced by H.

Suppose the letters $b_1, b_2, \cdots, b_k$ ($k \geq j$), new to M in the constituent of G_2 by which S replaces M, are permuted only among themselves by G_1. Clearly $k < m-1$, for if $k = m-1$, $\{G_1, G_2\}$ has a transitive constituent on the letters of M. Then $\{G_1, G_2\}$ is of degree $m+k+1 < 2m$. Of the letters $b_1, b_2, \cdots, b_k$, H displaces only $b_1, b_2, \cdots, b_j$ and therefore H_1 displaces only $a_1, a_2, \cdots, a_j$. The subgroup $\{H_1, H_1', \cdots, H_1^{m-1}\}$, invariant in G_1, displaces only letters of M. Being a subgroup of G of degree $<n$, it must be intransitive and therefore M, in which it is invariant, is imprimitive. If $k = j$, all the non-invariant subgroups of G_1, similar to H, are in a single conjugate set and are permuted according to the permutations of M. Each of these subgroups H_1, $H_1', \cdots, H_1^{m-1}$ is transformed into itself by H, and Theorem II is applicable to the group $\{H_1, H_1', \cdots\}$, which however is seen to fix the wrong letters. Then $k > j$.

Let it be assumed that M, *if imprimitive, is of degree $\leq n/2$.* Another assumption that might be made is that $k = j$. This last is a weaker form of Dr. Bennett's condition upon the degree of H: that it is $n-m-1$. It follows that there is in F at least one transitive constituent (B_1) on l of the k letters $b_1, b_2, \cdots$ which is a part of a transitive constituent P of G_1 in which occur letters $c, \cdots$ new to $S^{-1}MS$. Finally put upon G_1 the condition that these l letters of B_1 are permuted transitively by H. They may now be called $b_1, b_2, \cdots, b_l$ ($1 < l \leq j$). This transitive constituent P of G_1 is imprimitive because of its invariant intransitive subgroup in H.

If M is primitive, F is a maximal subgroup of G_1, and therefore F is the largest subgroup of G_1 by which the letters $b_1, b_2, \cdots, b_l$ are permuted only among themselves. There are m conjugate subgroups $F, F_1, \cdots$ in G_1. Hence P permutes m systems of imprimitivity, of which $b_1, b_2, \cdots b_l$ is one, according to the primitive group M.

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[*] Rietz, loc. cit., p. 9, Theorem 7.

If M is imprimitive, F is not maximal, and the letters b_1, b_2, $\cdots$, b_l may be transformed among themselves by a subgroup of G_1 of which F is a subgroup. Hence our transitive constituent has m' (a divisor of m) systems of imprimitivity of l letters each. These last results may be formulated as follows. The notation of this section is used.

THEOREM VI. *Let G_1 have a transitive constituent M, of order $<g/n$, paired with itself, and of degree $\leq n/2$ if M is imprimitive. There is a transitive constituent B_1 in $S^{-1}M_1S$ on l letters new to M which is a part of a transitive constituent P of G_1 in which are letters new to M and to $S^{-1}MS$. If the letters of B_1 are permuted transitively by H, P has m systems of imprimitivity of l letters each if M is primitive, or m' ($m' > 1$ and a divisor of m) systems of l letters each if M is imprimitive.*

For example, if all the transitive constituents of M_1 are primitive groups and if H_1 displaces $m-1$ or $m-2$ letters of M, then certainly the letters of B_1 are permuted transitively by H.

By putting on the restriction that $n \geq 2m$ when M is imprimitive, and with no corresponding condition when M is primitive, the condition "if no transitive constituent of degree l occurs in G_1, where l represents the degree of any one of the transitive constituents of the subgroup of M composed of all the substitutions leaving one letter of M fixed" of Dr. Bennett's Theorems V and VI, has been avoided. In the following theorem this condition is restored in a modified form.

THEOREM VII. *Let G_1 have a transitive constituent M of order $<g/n$, paired with itself. Let those transitive constituents of M_1 whose j letters are displaced by H_1 be primitive groups. Let G_1 have no constituent of degree j. Then G_1 has an imprimitive constituent of degree $m'l$, where l is the degree of one of the transitive constituents of M_1 whose letters are displaced by H_1 and where m' divides m if M is imprimitive and is equal to m if M is primitive.*

It was seen that H_1 displaces j letters a_1, a_2, $\cdots$, a_j of M_1 and that H displaces b_1, b_2, $\cdots$, b_j and no other letters of $S^{-1}M_1S$. Since H_1 is an invariant subgroup of F, H_1 displaces all the letters of a transitive constituent of M_1 if it displaces one of the letters of that constituent. If G_1 does not permute b_1, b_2, $\cdots$, b_j in one or several transitive constituents, one primitive constituent b_1, b_2, $\cdots$, b_l of F has a transitive subgroup in H and is part with letters c, $\cdots$, new to M and to $S^{-1}MS$, of an imprimitive constituent P of degree $m'l$. Our theorem follows as before.

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THE EXPANSION PROBLEMS ASSOCIATED WITH REGULAR DIFFERENTIAL SYSTEMS OF THE SECOND ORDER*

BY

M. H. STONE

In 1908 Birkhoff defined a class of series generalizing the Fourier and Sturm-Liouville series which arise from ordinary linear homogeneous differential equations of the second order with linear homogeneous boundary conditions. He starts from a differential system of order n with boundary conditions restricted to be regular and proves that an arbitrary function with a sufficient number of derivatives can be represented in terms of the solutions of this differential system.† Subsequently, Tamarkin extended the discussion to apply to functions integrable in the sense of Lebesgue by the device of comparing the formal Fourier and Birkhoff series for the same function.‡ Recently the author has published further researches on the problem.§

It is now proposed to complete in some respects our knowledge of regular differential systems of the second order. In our paper cited above, we left certain necessary conditions unstudied and now desire to fill this gap in the theory.‖ In addition, the attempt to discuss the formal expansions for functions integrable in accordance with definitions of integration more extended than that of Lebesgue has led us to results quite different from those hitherto established. We shall indicate the facts for functions integrable in the sense of Denjoy. The situation may be described roughly by saying that, whereas all the regular expansions associated with an arbitrary function integrable in the sense of Lebesgue have virtually the same behavior, the series formed for an arbitrary function integrable in the sense of Denjoy or non-absolutely integrable in the sense of Riemann fall into a non-denumerable infinity of distinct types exhibiting individual behavior. The result is due essentially to the fact that the Riemann-Lebesgue theorem does not hold true for non-absolutely integrable functions. On the other hand, if the series for an arbi-

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trary function integrable in the sense of Denjoy are considered with regard to their summability by means of sufficiently high order, this difference in behavior is found to disappear.

I. Preliminary lemmas

We shall need a number of lemmas for our later work, and prefer to collect them in one spot for reference, though none of them is essentially new. In several places we refer to other papers for the proofs.

Lemma I. *The differential equation*

$$u'' + (\rho^2 + g)u = 0, \qquad\qquad 0 \leqq x \leqq 1,$$

where ρ is a complex parameter and $g(x)$ an integral in the sense of Lebesgue, has two linearly independent solutions u_1, u_2, analytic in ρ, which together with their first derivatives can be put in the asymptotic forms

$$u_1 = e^{\rho i x}(1 + A/(\rho i) + O(1)/\rho^2), \qquad\qquad A = -\frac{1}{2}\int_0^x g\,dx,$$

$$u_1' = \rho i e^{\rho i x}(1 + A/(\rho i) + O(1)/\rho^2),$$

$$u_2 = e^{-\rho i x}(1 - A/(\rho i) + O(1)/\rho^2),$$

$$u_2' = -\rho i e^{-\rho i x}(1 - A/(\rho i) + O(1)/\rho^2),$$

valid $0 \leqq x \leqq 1$, $\Im(\rho) \geqq -c$, $c \geqq 0$. If $W(u_1, u_2)$ denotes the Wronskian of the two functions u_1 and u_2, then the functions $v_1 = -u_2/W$, $v_2 = +u_1/W$ and their first derivatives have the asymptotic forms

$$v_1 = e^{-\rho i x}(1 - A/(\rho i) + O(1)/\rho^2)/(2\rho i),$$

$$v_1' = -e^{-\rho i x}(1 - A/(\rho i) + O(1)/\rho^2)/2,$$

$$v_2 = -e^{\rho i x}(1 + A/(\rho i) + O(1)/\rho^2)/(2\rho i),$$

$$v_2' = -e^{\rho i x}(1 + A/(\rho i) + O(1)/\rho^2)/2,$$

valid on the same range.

The part of the lemma concerning the functions u_1 and u_2 is a special case of a general theorem due to Birkhoff. With the aid of Birkhoff's paper and a few comments upon it contained in a later one by the author, the truth of the statement is readily perceived.* The second part of the lemma, concerning the functions v_1 and v_2, is obtained by direct substitution.

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* For the proofs see Birkhoff, these Transactions, vol. 9 (1908), pp 219-231, and S, §I. The lemma is also a special case of a theorem of Tamarkin, T_2, Theorem 5.

LEMMA II. *If Γ_n is a semicircle of radius R_n with the origin as center, lying on the half-plane $\Im(\rho) \geq 0$, then*

$$\int_{\Gamma_n} (O(1)/\rho)d\rho = O(1).$$

LEMMA III. *If $0 < a \leq x \leq b < 1$, then*

$$\int_{\Gamma_n} \rho^k e^{\rho i x}(1 - \rho^2/R_n^2)^{k+l}O(1)d\rho = O(R_n^{-l}), \qquad k \geq 0, l \geq 0,$$

and

$$\int_{\Gamma_n} \rho^k e^{\rho i(1-x)}(1 - \rho^2/R_n^2)^{k+l}O(1)d\rho = O(R_n^{-l}), \quad k \geq 0, l \geq 0.$$

LEMMA IV. *The integral $\int_{\Gamma_n} e^{\rho i} O(1) \, d\rho$ is $O(1)$.*

LEMMA V. *If $0 \leq \alpha \leq x \leq 1$, then*

$$\int_\alpha^x \int_{\Gamma_n} (e^{\rho i(x-y)}O(1)/\rho)d\rho \, dy = o(1),$$

and, if $0 \leq x \leq \alpha \leq 1$, then

$$\int_\alpha^x \int_{\Gamma_n} (e^{\rho i(y-x)}O(1)/\rho)d\rho \, dy = o(1),$$

uniformly on the ranges considered.

Lemmas II, III, V are in essence Lemmas III′, V′, VI′ of our paper cited above as S; we refer to those lemmas for proofs. Lemma IV is obviously little different from the first part of Lemma III, with $x = 1$ and $k = l = 0$.

LEMMA VI. *If $\delta(\rho)$ is an analytic function with the asymptotic form*

$$[\theta_1]e^{2\rho i} + [\theta_0]e^{\rho i} + [\theta_2], \qquad \theta_1\theta_2 \neq 0,$$

on the region $\Im(\rho) \geq -c$, then the equation $\delta = 0$ has infinitely many roots in that region, distributed asymptotically near the roots of the equation

$$\theta_1 e^{2\rho i} + \theta_0 e^{\rho i} + \theta_2 = 0.$$

If these points are removed from the ρ-plane by deleting the interior of a small circle of preassigned radius described about each of them as center, the function $1/\delta$ is $O(1)$ in the remaining portion of the region $\Im(\rho) \geq -c$. The region obtained when $c = 0$ we denote by Σ'.[*]

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* B_1, p. 393; B_2, pp. 120–121; T_1, p. 353. The notation $[a]$ means as usual $a + O(1)/\rho$.

LEMMA VII. *If δ and $\bar{\delta}$ are functions of the type described in Lemma VI, with $\theta_0 = \bar{\theta}_0$, $\theta_1 = \bar{\theta}_1$, $\theta_2 = \bar{\theta}_2$, then on Σ' we have $\delta/\bar{\delta} = [1]$.* *

LEMMA VIII. *If $F(x, y)$ is continuous for x and y on the interval (a, b), then*

$$\int_a^b F(x,y) \sin R(x - y)dy = o(1)$$

uniformly $a \leq x \leq b$ as R becomes infinite.†

We can approximate to F by a function F_1 constant on each rectangle of a network fitted on the square $a \leq x \leq b$, $a \leq y \leq b$, so that

$$\left| \int_a^b (F - F_1) \sin R(x - y)dy \right| \leq \epsilon/2.$$

Then for R sufficiently great

$$\left| \int_a^b F_1 \sin R(x - y)dy \right| \leq \epsilon/2.$$

Consequently,

$$\left| \int_a^b F \sin R(x - y)dy \right| \leq \epsilon$$

when R is taken large enough, as we wished to show.

LEMMA IX. *If we denote by k a fixed positive integer; by c_i, $i = 1, \cdots, k$, a set of distinct constants, real or complex; and by P_{in}, Q_{in}, $i = 1, \cdots, k$, a set of polynomials of degree less than or equal to m independent of n, for $n = 1$, $2, 3, \cdots$, with real or complex coefficients—then a necessary and sufficient condition that as n becomes infinite*

$$\sum_{i=1}^k (P_{in} \cos (2n\pi + c_i)x + Q_{in} \sin (2n\pi + c_i)x) = o(1)$$

on a fixed interval (a, b) is that the coefficients in the polynomials be $o(1)$.

The sufficiency of the condition is obvious. To prove its necessity we let s_n represent the sum of the absolute values of the coefficients in the polynomials P and Q, and then derive a contradiction from the supposition that there can be found an infinite sequence of values of n for which the quantities s_n have a positive lower bound. If such a sequence exists, it is clear that we can determine a subsequence thereof for which

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* S, Lemmas VIII and VIII'.

† For general results of this character, see Langer and Tamarkin, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 335–337.

$$P_{in}/s_n = P_i + o(1), \qquad Q_{in} = Q_i + o(1) \quad (i = 1, \cdots, k).$$

On dividing the expression given in the statement of the lemma by s_n we see that

$$\sum_{i=1}^{k} (P_i \cos (2n\pi + c_i)x + Q_i \sin (2n\pi + c_i)x) = o(1)$$

for the values of n which we have retained. The real and imaginary parts of this equation can both be put in the form

$$A(x) \cos 2n\pi x + B(x) \sin 2n\pi x = o(1)$$

where $A^2 + B^2$ is not identically zero in one of the two equations so obtained. By squaring the last equation and performing some simple trigonometric reductions it becomes

$$A^2 + B^2 + (A^2 - B^2) \cos 4n\pi x + 2AB \sin 4n\pi x = o(1).$$

Since A and B are continuous functions in view of their origin, the function on the left in this equation is bounded and its integral over (a, b) approaches zero as n becomes infinite; but, by the theorem of Riemann-Lebesgue, this integral has the limit $\int_a^b (A^2 + B^2) dx$, which is different from zero unless $A^2 + B^2$ vanishes identically. This gives us the desired contradiction; we must have $s_n = o(1)$ and can thus establish the lemma at once.

LEMMA X. *Let $\{n_p\}$ be a sequence of integers such that $\sum_{p=1}^{\infty} p^2/\log n_p$ converges, n_{p+1} is greater than $2n_p$, and $n_p/(p \log n_p)$ becomes infinite with p.* Then the function defined by the equations*

$$\phi(0) = \phi(1) = 0,$$

$$\phi(x) = (2p^2 n_p/\log n_p) \sin 2n_p\pi x, \quad 1/(p + 1) \leqq x < 1/p \ (p = 1, 2, \cdots),$$

is integrable in the sense of Denjoy on $(0, 1)$; and, if c is any constant,

$$\int_0^1 \phi(y) \sin (2n_q\pi + c)y \, dy = (n_q/\log n_q)(1 + O(1/q)) + O(1),$$

$$\int_0^1 \phi(y) \cos (2n_q\pi + c)y \, dy = (-cn_q/\log n_q)(1 + O(1/q)) + O(1),$$

$$\int_0^1 \phi(y) \, y \sin 2n_q\pi y \, dy = (n_q/(q \log n_q))(1 + O(1/q)) + O(1),$$

$$\int_0^1 \phi(y) y \cos 2n_q\pi y \, dy = O(1).$$

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* We might, for instance, take n_p as the integral part of exp p^4. It is clear that $n_{p+1} > 2n_p$ and that $\log n_p = p^4 + O(1)$. The other properties follow immediately.

The function $\phi(x)$ is integrable in the sense of Denjoy, and non-absolutely integrable in the sense of Riemann, since the series

$$\sum_{p=1}^{\infty} \int_{1/(p+1)}^{1/p} \phi(y)dy = \sum_{p=1}^{\infty} p^2 O(1)/\log n_p$$

is convergent. The four integrals involving ϕ can be investigated by actually performing the integration in each case. We treat the first of them as typical, omitting the proofs for the other three. Writing I_p for the contribution to the integral from the interval $(1/(p+1), 1/p)$ we find

$$I_p = \int_{1/(p+1)}^{1/p} \phi(y) \sin (2n_q\pi + c)y\, dy$$

$$= (p^2/\log n_p) \int_{1/(p+1)}^{1/p} (- \cos (2(n_p + n_q)\pi + c)y$$
$$+ \cos (2(n_p - n_q)\pi - c)y)dy.$$

Thus, when p and q are different,

$$I_p = p^2 n_p O(1)/((\log n_p)(2(n_p + n_q)\pi + c))$$
$$+ p^2 n_p O(1)/((\log n_p)(2(n_p - n_q)\pi - c)).$$

From the requirement that $n_{p+1} > 2n_p$ it is apparent that

$$n_p/(2(n_p + n_q)\pi + c) = O(1), \quad n_p/(2(n_p - n_q)\pi - c) = O(1),$$

whence $I_p = p^2 O(1)/\log n_p$, $p \neq q$. When p and q are equal

$$I_q = (q^2/\log n_q)O(1) + (\sin (c/q) - \sin(c/(q + 1)))q^2 n_q/(c \log n_q).$$

Now by Taylor's theorem with remainder we have

$$(\sin (c/q) - \sin(c/(q + 1)))/c = 1/q^2 + O(1/q^3).$$

In consequence,

$$I_q = q^2/\log n_q + (n_q/\log n_q)(1 + O(1/q)).$$

Finally, the convergence of the infinite series whose general term is $p^2/\log n_p$ establishes the desired result

$$\sum_{p=1}^{\infty} I_p = (n_q/\log n_q)(1 + O(1/q)) + O(1).$$

The proof here is due to Titchmarsh.[*]

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[*] Titchmarsh, Proceedings of the London Mathematical Society, (2), vol. 22 (1923–1924), pp. xxv–xxvi.

II. The expansions of summable functions

We wish to investigate the formal expansions of a function integrable in the sense of Lebesgue on the interval $(0, 1)$ in terms of the solutions of the differential system

$$u'' + (\rho^2 + g)u = 0, \qquad\qquad 0 \le x \le 1,$$

$$W_1(u) = \alpha_1 u^{(k_1)}(0) + \beta_1 u^{(k_1)}(1) + \cdots = 0,$$

$$W_2(u) = \alpha_2 u^{(k_2)}(0) + \beta_2 u^{(k_2)}(1) + \cdots = 0,$$

$$2 > k_1 \ge k_2 \ge 0,$$

where $g(x)$ is summable on $(0, 1)$, and the boundary conditions W_1 and W_2 are reduced to normal form and are regular.* There is defined by this differential system a Green's function $G(x, y; \rho^2)$ meromorphic in the ρ-plane. The poles of G on the region $\Im(\rho) \ge -c$ occur at the roots of an equation of the form $\delta(\rho) = 0$, where δ is of the character described in Lemma VI. On the region Σ' determined by the function δ we define a sequence of semicircles Γ_n with centers at $\rho = 0$ and radii R_n such that the half-ring between Γ_n and Γ_{n+1} contains at least one pole of G and as few others as possible. The numbers R_n increase monotonely to $+\infty$. The formal expansion of a function $f(x)$ can then be expressed as

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} 2\rho G(x, y; \rho^2) d\rho \, dy \,;$$

and means of order l similar to the Riesz typical means of order l for this expression can then be studied with the aid of the expression

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} 2\rho(1 - \rho^2/R_n^2)^l G(x, y; \rho^2) d\rho \, dy, \ l > 0.$$

For fuller discussions of these facts we refer to other papers.†

If we wish to examine simultaneously two differential systems of this type with Green's functions G and $\overline{G}$, we can choose one sequence of semicircles Γ_n lying on the region common to Σ' and $\overline{\Sigma}'$, effective for both systems. In fact, the distribution of the poles of the two Green's functions, that is, of the roots of the two equations $\delta = 0$, $\overline{\delta} = 0$, is such that the semicircles are merely restricted to lie in a certain sequence of half-rings centered at the origin. This appears at once from Lemma VI.

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* For the terms normal and regular, see B_1, pp. 382–383.

† B_1, pp. 373–380; S, §§II, III.

If in Theorem XIX′ of our paper cited as S we take $n = 2$ and combine the results thus obtained for the two quadrants $0 \leqq \arg \rho \leqq \pi/2$, $\pi/2 \leqq \arg \rho \leqq \pi$, we find

THEOREM I. *If $f(x)$ is any function summable on the interval $(0, 1)$, then the integral*

$$\int_0^1 f(y) \int_{\Gamma_n} (2\rho G - \{ -ie^{\rho i(x-y)}\ ;\ -ie^{\rho i(y-x)} \} - \mathcal{D}(x, y\ ;\ \rho)/\theta_2) d\rho\, dy$$

is $o(1)$ uniformly $0 \leqq x \leqq 1$. The constant θ_2 has the value

$$i^{k_1+k_2}(\alpha_1\beta_2(-1)^{k_2} - \alpha_2\beta_1(-1)^{k_1})$$

and the function $\mathcal{D}$ the expression

$$\begin{vmatrix} e^{\rho i x} & e^{\rho i(1-x)} & 0 \\ \alpha_1 i^{k_1} & \beta_1(-i)^{k_1} & -\beta_1 i^{k_1+1} e^{\rho i(1-y)} + \alpha_1(-i)^{k_1+1} e^{\rho i y} \\ \alpha_2 i^{k_2} & \beta_2(-i)^{k_2} & -\beta_2 i^{k_2+1} e^{\rho i(1-y)} + \alpha_2(-i)^{k_2+1} e^{\rho i y} \end{vmatrix}^{*}.$$

From this theorem we derive

THEOREM II. *A necessary and sufficient condition that two differential systems of the second order with boundary conditions reduced to normal form and restricted to be regular be such that*

$$I_n = \frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} 2\rho(G - \overline{G}) d\rho\, dy = o(1)$$

uniformly $0 \leqq x \leqq 1$ for every summable function $f(x)$ is that the constants α, β, k of the boundary conditions satisfy one of the three relations

(1)　　$k_1 = k_2 = \bar{k}_1 = \bar{k}_2 = 1$;

(2)　　$k_1 = k_2 = \bar{k}_1 = \bar{k}_2 = 0$;

(3)　　$k_1 = \bar{k}_1 = 1$,　　$k_2 = \bar{k}_2 = 0$,　　$\bar{\alpha}_1 = \lambda\alpha_1$,　　$\bar{\beta}_1 = \lambda\beta_1$,　　$\bar{\alpha}_2 = \mu\alpha_2$,　　$\bar{\beta}_2 = \mu\beta_2$

where λ and μ are constants different from zero.

By Theorem I,

$$2\pi i I_n = \int_0^1 f(y) \int_{\Gamma_n} ((\overline{A} - A)e^{\rho i(x+y)} + (\overline{B} - B)e^{\rho i(x+1-y)}$$
$$+ (\overline{C} - C)e^{\rho i(1-x+y)} + (\overline{D} - D)e^{\rho i(2-x-y)}) d\rho\, dy$$

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* See S, Theorem XIX′. For the evaluation of the constant θ_2, compare B_1, p. 383. The notation $\{A\ ;B\}$ is employed to indicate a function reducing to A when $x \geqq y$ and to B when $x < y$.

where the coefficients of the exponentials in the integral are constants depending only on the boundary conditions. By the theory of contour integration

$$\int_{\Gamma_n} e^{\rho i \omega} d\rho = \int_{R_n}^{-R_n} e^{\rho i \omega} d\rho = -2 \sin (R_n \omega)/\omega,$$

so that the explicit expression on the right of the equation above may be written

$$\int_0^1 f(y)\phi(x, y ; n)dy,$$

$$\phi(x, y ; n) = 2((A - \overline{A})\psi_n(x + y) + (B - \overline{B})\psi_n(1 + x + y)$$
$$+ (C - \overline{C})\psi_n(1 - x + y) + (D - \overline{D})\psi_n(2 - x - y)),$$

$$\psi_n(\omega) = \sin (R_n \omega)/\omega.$$

A necessary and sufficient condition that I_n be $o(1)$ uniformly $0 \leq x \leq 1$ is that $\int_0^1 f(y) \phi(x, y; n)dy = o(1)$ uniformly on the same range. By a well known theorem of Lebesgue a necessary condition that the last equation be true is that the function ϕ be uniformly bounded, $0 \leq x \leq 1, 0 \leq y \leq 1$.* Clearly this is impossible unless $A = \overline{A}$, $B = \overline{B}$, $C = \overline{C}$, $D = \overline{D}$; that is, unless ϕ is identically zero. Consequently these four equations constitute a necessary and sufficient condition that I_n be $o(1)$ uniformly $0 \leq x \leq 1$. It can be shown algebraically that these equations imply one or another of the three sets of equations enumerated in the statement of the theorem and conversely. This establishes the theorem.†

It is clear that the method employed could be extended to differential systems of arbitrary order with regular boundary conditions, since general theorems of which our Theorem I is a special case have been established for them. The algebraic difficulties would be considerable in the final stages of such a discussion.

III. The expansions for totalisable functions

The formal expansion of a function integrable in the sense of Denjoy in terms of the solutions of a regular differential system of the second order is

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* Lebesgue, Annales de la Faculté des Sciences de Toulouse, (3), vol. 1 (1909), pp. 52–55.

† The sufficiency of the conditions given by the theorem can be proved by some results of Tamarkin, T_2, pp. 104–112, and has been proved explicitly by the author, S, Theorem XVIII'. The necessity of the conditions can be deduced by the use of a theorem of Tamarkin concerning the uniform convergence of the Birkhoff series for a function of bounded variation, T_2, Theorem 14; the generalisation of the present theorem to Birkhoff series of arbitrary order could not be effected in that manner.

given by the expressions of the preceding section. We shall restrict the differential equation $u'' + (\rho^2 + g)u = 0$ by the requirement that $g(x)$ be an integral in the sense of Lebesgue, $0 \leq x \leq 1$. We first prove

THEOREM III. *If $f(x)$ is totalisable, $0 \leq x \leq 1$, and if $\phi(x, y; \rho)$ denotes the function*

$$\frac{\begin{vmatrix} e^{\rho i x} & e^{\rho i(1-x)} & 0 \\ \alpha_1 i^{k_1} + \beta_1 i^{k_1} e^{\rho i} & \alpha_1(-i)^{k_1} e^{\rho i} + \beta_1(-i)^{k_1} & -\beta_1 i^{k_1+1} e^{\rho i(1-y)} + \alpha_1(-i)^{k_1+1} e^{\rho i y} \\ \alpha_2 i^{k_2} + \beta_2 i^{k_2} e^{\rho i} & \alpha_2(-i)^{k_2} e^{\rho i} + \beta_2(-i)^{k_2} & -\beta_2 i^{k_2+1} e^{\rho i(1-y)} + \alpha_2(-i)^{k_2+1} e^{\rho i y} \end{vmatrix}}{\theta_2 + \theta_0 e^{\rho i} + \theta_1 e^{2\rho i}}$$

then the integral

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (2\rho G - \{ -ie^{\rho i(x-y)}; -ie^{\rho i(y-x)}\}$$

$$-\phi(x, y; \rho))d\rho\, dy$$

is $o(1)$, $l \geq 0$, uniformly $0 < a \leq x \leq b < 1$.

The explicit formula for the Green's function is

$$G = \{u_1(x)v_1(y); -u_2(x)v_2(y)\} + \psi(x, y; \rho),$$

$$\psi = \begin{vmatrix} u_1(x) & u_2(x) & 0 \\ W_1(u_1) & W_1(u_2) & W_{11}(u_1)v_1(y) - W_{10}(u_2)v_2(y) \\ W_2(u_1) & W_2(u_2) & W_{21}(u_1)v_1(y) - W_{20}(u_2)v_2(y) \end{vmatrix} \div \begin{vmatrix} W_1(u_1) & W_1(u_2) \\ W_2(u_1) & W_2(u_2) \end{vmatrix}$$

where u_1, u_2, v_1, v_2 are the functions described in Lemma I, and W_{k0} and W_{k1} are the parts of the boundary condition W_k involving $x = 0$ and $x = 1$ respectively.[*] We first study the integral

$$I_n = \int_0^1 f(y) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (2\rho\{u_1(x)v_1(y); -u_2(x)v_2(y)\}$$

$$- \{-ie^{\rho i(x-y)}; -ie^{\rho i(y-x)}\})d\rho\, dy.$$

By the formula for integration by parts applicable to the integral with respect to y, we find

$$I_n = F(1) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (ie^{\rho i(1-x)} - 2\rho u_2(x)v_2(1))d\rho$$

$$- \int_0^1 F(y) \int_{\Gamma_n} \chi(x, y; \rho)d\rho\, dy,$$

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[*] B_1, p. 378, p. 392; T_1, p. 363.

where

$$F(x) = \int_0^x f\, dy,$$

$$\chi(x,y\;;\rho) = (1 - \rho^2/R_n^2)^l(2\rho\{u_1(x)v_1'(y)\;;\; -u_2(x)v_2'(y)\}$$
$$- \rho\{-e^{\rho i(x-y)}\;;\; e^{\rho i(y-x)}\}).^*$$

The first term on the right, when the asymptotic forms for u_2 and v_2 are substituted from Lemma I, is seen to be of the form

$$\int_{\Gamma_n}((1 - \rho^2/R_n^2)^l e^{\rho i(1-x)}O(1)/\rho)d\rho = \int_{\Gamma_n}e^{\rho i(1-x)}O(1)d\rho/R_n$$

and by Lemma III is $o(1)$ uniformly $0 < a \leq x \leq b < 1$. The second term becomes

$$\chi = (A(x) - A(y))i(1 - \rho^2/R_n^2)^l\{e^{\rho i(x-y)}\;;\; e^{\rho i(y-x)}\}$$
$$+ (1 - \rho^2/R_n^2)^l\{e^{\rho i(x-y)}O(1)/\rho\;;\; e^{\rho i(y-x)}O(1)/\rho\}.$$

We denote the terms on the right by χ_1 and χ_2 respectively. Then

$$\int_0^1 F(y)\int_{\Gamma_n}\chi_1 d\rho\, dy = \int_0^1 F(y)\int_{R_n}^{-R_n}\chi_1 d\rho\, dy$$
$$= 2\int_0^{R_n}(1 - \rho^2/R_n^2)^l\int_0^1 F(y)(A(x) - A(y))\cos\rho\,(x - y)\, dy\, d\rho.$$

This integral has the same behavior when n becomes infinite as the integral with $l=0$, of which it is a Riesz sum.[†] The integral with $l=0$ is seen by Lemma VIII to be $o(1)$ uniformly, $0 \leq x \leq 1$; for, when the integration with respect to ρ is performed, this integral can be written as

$$2\int_0^1 (F(y)(A(x) - A(y))/(x - y))\sin R_n(x - y)\, dy.$$

Next we discuss the integral involving χ_2 by the aid of the theorem of Lebesgue cited in connection with Theorem II. Since $\chi_2 = O(1)/\rho$ it follows from Lemma II that $\int_{\Gamma_n}\chi_2 d\rho = O(1)$ uniformly, $0 \leq x \leq 1$; and from Lemma V

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* The use of integration by parts is a standard device in the consideration of the Fourier series of totalisable functions. See for instance Nalli, Rendiconti del Circolo Matematico di Palermo, vol. 40 (1915), pp. 33–37, and Hobson, Proceedings of the London Mathematical Society, (2), vol. 22 (1924), pp. 420–424.

† Hardy and Riesz, *The General Theory of Dirichlet Series* (Cambridge Tracts, No. 18, 1915), Chapters IV and V.

we see that $\int_\alpha^\beta \int_{\Gamma_n} \chi_2 \, d\rho \, dy = o(1)$ uniformly, $0 \leq x \leq 1, 0 \leq \alpha \leq \beta \leq 1$. Lebesgue's theorem now yields the result $\int_0^1 F(y) \int_{\Gamma_n} \chi_2 \, d\rho \, dy = o(1)$ uniformly, $0 \leq x \leq 1$. Combining these results we see than I_n is also $o(1)$ uniformly on the same range.

It remains for us to show that

$$J_n = \int_0^1 f(y) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (\psi - \phi) d\rho \, dy = o(1)$$

uniformly, $0 < a \leq x \leq b < 1$. An integration by parts yields the result

$$J_n = F(1) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (\psi(x, 1 ; \rho) - \phi(x, 1 ; \rho)) d\rho$$

$$- \int_0^1 F(y) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \frac{\partial}{\partial y} (\psi - \phi) d\rho \, dy.$$

If in the expressions $\psi - \phi$, $(\partial/\partial y)(\psi - \phi)$ we substitute the asymptotic expansions from Lemma I, expand the determinants involved, and evaluate the coefficients of the various exponentials occurring therein by the aid of Lemmas VI and VII, we find

$$\psi - \phi = e^{\rho i x} O(1)/\rho + e^{\rho i (1-x)} O(1)/\rho,$$

$$\frac{\partial}{\partial y}(\psi - \phi) = e^{\rho i x} O(1) + e^{\rho i (1-x)} O(1),$$

for $0 \leq x \leq 1, 0 \leq y \leq 1$, and for all ρ on Σ'.* Consequently,

$$\int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (\psi - \phi) d\rho = \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (e^{\rho i x} O(1) + e^{\rho i (1-x)} O(1)) d\rho/R_n$$

is $o(1)$ uniformly, $0 < a \leq x \leq b < 1, 0 \leq y \leq 1$, by Lemma III. We make twofold use of this fact. In the first place, it shows that the first term in J_n is $o(1)$ uniformly on (a, b). In the second place, it provides us with one of the two sufficient conditions of Lebesgue's theorem when we apply it to the second term in J_n; for it proves that

$$\int_\alpha^\beta \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \frac{\partial}{\partial y}(\psi - \phi) d\rho \, dy$$

is $o(1)$ uniformly on (a, b). The other condition of that theorem is seen to be satisfied when we use Lemma III to prove that

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*A somewhat more detailed discussion of a similar question is to be found in S, Theorem XV.

$$\int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \frac{\partial}{\partial y}(\psi - \phi)d\rho$$

is $O(1)$ uniformly, $0 < a \leq x \leq b < 1$, $0 \leq y \leq 1$. Thus we see that the second term in J_n is $o(1)$ uniformly on (a, b).

Thus we have shown that $I_n + J_n$ is $o(1)$ uniformly on (a, b); this is the assertion of the theorem.

THEOREM IV. *If $f(x)$ is totalisable, $0 \leq x \leq 1$, and if $l \geq 1$, then*

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l (2\rho G(x, y ; \rho^2) - \{ - ie^{\rho i(x-y)} ; \; - ie^{\rho i(y-x)} \}) d\rho \, dy$$

is $o(1)$ uniformly, $0 < a \leq x \leq b < 1$. Thus if G and $\overline{G}$ are the Green's functions for two regular differential systems of the second order, and if the sequence of semicircles Γ_n is chosen to apply to both, then

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} 2\rho(1 - \rho^2/R_n^2)^l (G - \overline{G}) d\rho \, dy = o(1), \quad l \geq 1,$$

uniformly on the same range. In particular $\overline{G}$ may be chosen as the Green's function of the Fourier differential system

$$u'' + \rho^2 u = 0, \qquad u(0) - u(1) = u'(0) - u'(1) = 0 ;$$

consequently, the behavior of the sum of order $l \geq 1$ for an arbitrary expansion of the class considered is the same as that of a sum of order l for the corresponding Fourier series.

Since on Σ'

$$1/(\theta_2 + \theta_0 e^{\rho i} + \theta_1 e^{2\rho i}) = 1/\theta_2 - (\theta_0 e^{\rho i} + \theta_1 e^{2\rho i})/(\theta_2(\theta_2 + \theta_0 e^{\rho i} + \theta_1 e^{2\rho i}))$$

$$= 1/\theta_2 + e^{\rho i} O(1),$$

we can write

$$\phi(x, y ; \rho) = A e^{\rho i(x+y)} + B e^{\rho i(x+1-y)} + C e^{\rho i(1-x+y)} + D e^{\rho i(2-x-y)} + e^{\rho i} O(1),$$

where the coefficients of the exponentials are the constants defined in the course of the proof of Theorem II. In the same way

$$\frac{\partial \phi}{\partial y} = A\rho i e^{\rho i(x+y)} - B\rho i e^{\rho i(x+1-y)} + C\rho i e^{\rho i(1-x+y)} - D\rho i e^{\rho i(2-x-y)} + \rho e^{\rho i} O(1).$$

Lemma III then shows that

$$\int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \phi \, d\rho = o(1), \qquad l \geqq 1,$$

$$\int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \frac{\partial \phi}{\partial y} \, d\rho = O(1), \qquad l \geqq 1,$$

uniformly, $0 < a \leqq x \leqq b < 1$, $0 \leqq y \leqq 1$. From the theorem of Lebesgue which we have used so frequently we conclude that

$$\int_0^1 f(y) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \phi \, d\rho \, dy = F(1) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \phi(x, 1 \,;\, \rho) d\rho$$

$$- \int_0^1 F(y) \int_{\Gamma_n} (1 - \rho^2/R_n^2)^l \frac{\partial \phi}{\partial y} \, d\rho \, dy = o(1), \quad l \geqq 1,$$

uniformly on (a, b). By combining this result with Theorem III the present theorem is established without difficulty.

THEOREM V. *If for an arbitrary function totalisable on* $(0, 1)$ *and for some fixed value of* k

$$I_n = \frac{1}{2\pi i} \int_0^1 f(y) \left(\int_{\Gamma_n} 2\rho G \, d\rho - \int_{\Gamma_{n+k}} 2\rho \overline{G} \, d\rho \right) dy = o(1)$$

uniformly or not on a fixed interval, however small, then it is necessary that the boundary conditions of the differential systems considered satisfy one of the relations (1)–(3) *of Theorem* II. *Conversely, if the boundary conditions of the two systems are so related, then*

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} 2\rho(G - \overline{G}) d\rho \, dy = o(1)$$

uniformly, $0 < a \leqq x \leqq b < 1$.

As a consequence of Theorem III, it is sufficient to consider differential systems of the form

$$u'' + \rho^2 u = 0,$$

$$\alpha_1 u^{(k_1)}(0) + \beta_1 u^{(k_1)}(1) = 0,$$

$$\alpha_2 u^{(k_2)}(0) + \beta_2 u^{(k_2)}(1) = 0,$$

$$2 > k_1 \geqq k_2 \geqq 0.$$

We shall evaluate I_n explicitly in terms of the solutions of two such systems.

There are three main cases to consider, according to the relations between k_1 and k_2 in the boundary conditions of such a system. In Case I, $k_1 = k_2 = 1$, the boundary conditions can be put in the form $u'(0) = u'(1) = 0$, and the formal expansions are cosine series on $(0, 1)$. In Case II, $k_1 = k_2 = 0$, the expansions are sine series on $(0, 1)$. In Case III, $k_1 = 1$, $k_2 = 0$, the expansions fall into three main types and two of these types are further subdivided into five sub-types each, as we shall now show.

In Case III we first compute the differential system adjoint to

$$u'' + \rho^2 u = 0, \quad \alpha_1 u'(0) + \beta_1 u'(1) = 0, \quad \alpha_2 u(0) + \beta_2 u(1) = 0,$$

as

$$v'' + \rho^2 v = 0, \quad \beta_2 v'(0) + \alpha_2 v'(1) = 0, \quad \beta_1 v(0) + \alpha_2 v(1) = 0.$$

These systems are found to have solutions for $\rho = 2n\pi + a_1$, $\rho = 2n\pi + a_2$, where $\cos a_1 = \cos a_2 = -(\alpha_1\alpha_2 + \beta_1\beta_2)/(\alpha_1\beta_2 + \alpha_2\beta_1)$, $0 \leq a_1 < 2\pi$, $0 \leq a_2 < 2\pi$. The condition that the boundary conditions be regular makes the denominator in this fraction different from zero. The three divisions of Case III are Type 1, $a_1 \neq a_2$; Type 2, $a_1 = a_2 = 0$; Type 3, $a_1 = a_2 = \pi$.

In Type 1,

$$
\begin{aligned}
u &= \alpha_2 \sin \rho x - \beta_2 \sin \rho(1 - x) \\
&= (\alpha_2 + \beta_2 \cos \rho) \sin \rho x - \beta_2 \sin \rho \cos \rho x \\
&= C_1 \sin \rho x + C_2 \cos \rho x, \\
v &= \beta_1 \sin \rho x - \alpha_1 \sin \rho(1 - x) \\
&= (\beta_1 + \alpha_1 \cos \rho) \sin \rho x - \alpha_1 \sin \rho \cos \rho x \\
&= D_1 \sin \rho x + D_2 \cos \rho x
\end{aligned}
$$

are the only solutions of the two systems corresponding to the same characteristic value ρ. For them

$$\int_0^1 uv \, dx = C_1 D_1 \int_0^1 \sin^2 \rho x \, dx + (C_1 D_2 + C_2 D_1) \int_0^1 \sin \rho x \cos \rho x \, dx$$

$$= C_2 D_2 \int_0^1 \cos^2 \rho x \, dx = 1/K + O(1/\rho) = 1/K + O(1/n)$$

where

$$1/K = (C_1 D_1 + C_2 D_2)/2 = (\alpha_1\beta_2 + \alpha_2\beta_1)(\sin^2 \rho)/2 \neq 0.$$

Thus the formal expansion for a function $f(x)$ in terms of the functions u is

$$\sum_{(n)} (K + O(1/n))(C_1 \sin \rho x + C_2 \cos \rho x) \int_0^1 f(y)(D_1 \sin \rho y + D_2 \cos \rho y) dy$$

where the sum is extended over the characteristic values of ρ for which arg ρ is greater than or equal to 0 and less than π.

In Type 2, $\cos \rho = 1$ is the characteristic equation, so that we must have $(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) = 0$. The characteristic values $\rho = 2n\pi$ are double roots of the characteristic equation, so that the character of the expansions must be determined by computing the residues for the Green's function of the differential system. Since the general solution of the differential equation $u'' + \rho^2 u = 0$ when $\rho = 2n\pi$ is a linear combination of $\cos 2n\pi x$ and $\sin 2n\pi x$ and satisfies the periodic boundary conditions $u'(0) - u'(1) = u(0) - u(1) = 0$, the conditions

$$\alpha_1 u'(0) + \beta_1 u'(1) = 0, \ \ \alpha_2 u(0) + \beta_2 u(1) = 0, \ \ (\alpha_1 + \beta_1)(\alpha_2 + \beta_2) = 0$$

must be compatible with them if $\rho = 2n\pi$ is to be a characteristic value for the differential system. Thus five cases are possible:

(i) $$u'(0) - u'(1) = u(0) - u(1) = 0 ;$$

(ii) $$u'(0) - u'(1) = u(0) = 0 ;$$

(iii) $$u'(0) - u'(1) = u(1) = 0 ;$$

(iv) $$u'(0) = u(0) - u(1) = 0 ;$$

(v) $$u'(1) = u(0) - u(1) = 0 .$$

The residues of the Green's functions in the various cases can be computed directly, and are found to be

(i) $$2 \cos 2n\pi(x - y) ;$$

(ii) $$4(1 - y) \sin 2n\pi x \sin 2n\pi y + 4x \cos 2n\pi x \cos 2n\pi y ;$$

(iii) $$4(1 - x) \cos 2n\pi x \cos 2n\pi y + 4y \sin 2n\pi x \sin 2n\pi y ;$$

(iv) $$4(1 - y) \cos 2n\pi x \cos 2n\pi y + 4x \sin 2n\pi x \sin 2n\pi y ;$$

(v) $$4(1 - x) \sin 2n\pi x \sin 2n\pi y + 4y \cos 2n\pi x \cos 2n\pi y.$$

The series of Type 2 therefore fall into five subtypes.

The consideration of Type 3 where the characteristic equation is $\cos \rho = -1$ is parallel to that of Type 2. The five sets of boundary conditions possible can be obtained from those given under Type 2 by replacing each minus sign by a plus sign; and the corresponding residues of the Green's functions can be found by replacing $2n$ by $2n+1$ in the residues given under Type 2.

We are now prepared to discuss the consequences of the hypothesis that I_n is $o(1)$ on a fixed interval. We form I_n for the function ϕ of Lemma X;

then we must have $I_{n+1} - I_n = o(1)$. If the two differential systems involved in I_n have characteristic values of different forms, then $I_{n+1} - I_n$ is an algebraico-trigonometric sum of the type described in Lemma IX; and each system contributes a trigonometric function not contributed by the other. By Lemma X there is at least one coefficient not $o(1)$ in this sum. Hence, according to Lemma IX, we arrive at the contradictory statement that $I_{n+1} - I_n$ cannot be $o(1)$. We may illustrate the discussion by considering two series under Case III, Type 1. We have

$$I_{n+1} - I_n$$

$$= \sum (K + O(1/n))(C_1 \sin \rho x + C_2 \cos \rho x) \int_0^1 \phi(y)(D_1 \sin \rho y + D_2 \cos \rho y)dy$$

$$- \sum (\overline{K} + O(1/n))(\overline{C}_1 \sin \overline{\rho} x + \overline{C}_2 \cos \overline{\rho} x) \int_0^1 \phi(y)(\overline{D}_1 \sin \overline{\rho} y + \overline{D}_2 \cos \overline{\rho} y)dy$$

where the first sum is extended over a limited number of values of ρ of the form $\rho = 2n\pi + 2m_1\pi + a_1$, $\rho = 2n\pi + 2m_2\pi + a_2$, and the second over a limited number of values of $\overline{\rho}$ of similar form. We have also $\cos \rho \neq \cos \overline{\rho}$. The integrals in this sum are not $o(1)$ by Lemma X. Thus $I_{n+1} - I_n$ cannot be $o(1)$ in this case.

It remains for us to consider those cases in which the two systems have characteristic values of the same form. There are four of these: the comparison of a series in Case I with the corresponding series in Case II; the comparison of series of different subtypes in Case III, Type 2; the comparison of series of different subtypes in Case III, Type 3; and the comparison of series in Case III, Type 1.

When the two series compared are in Cases I and II respectively the reasoning given above applies without modification, since one is a cosine, the other a sine series.

When the two series are of different subtypes under Case III, Type 2, each series contributes to the expression $I_{n+1} - I_n$ an algebraico-trigonometric term not contributed by the other, and if the function for which the series are formed is taken as the function ϕ of Lemma X at least one coefficient in $I_{n+1} - I_n$ is not $o(1)$. The desired result follows at once. For example, we consider series of subtypes (ii) and (iii). For the function ϕ the difference $I_{n+1} - I_n$ is seen to be

$$- 4(1 - 2x) \cos 2n\pi x \int_0^1 \phi(y) \cos 2n\pi y \, dy$$

$$+ 4 \sin 2n\pi x \int_0^1 \phi(y)(1 - 2y) \sin 2n\pi y \, dy$$

under the assumption that the integer k introduced in the hypothesis of the theorem is zero. This expression clearly is not $o(1)$. If the integer k were different from zero the same result would follow, since no function appearing in $I_{n+1} - I_n$ would come from both series.

When the two series are of different subtypes under Case III, Type 3, the reasoning of the preceding paragraph applies without change.

It remains for us to examine systems in Case III, Type 1, with the same characteristic values. Clearly the partial sums whose difference we denote by I_n must be such that $I_{n+1} - I_n$ involves the same trigonometric functions from both sums; that is, the integer k must be zero. Otherwise Lemmas IX and X would show that $I_{n+1} - I_n$, when formed for the function ϕ, could not be $o(1)$. Hence we are able to write

$$I_{n+1} - I_n$$

$$= \sum (K + O(1/n))(C_1 \sin \rho x + C_2 \cos \rho x) \int_0^1 \phi(y)(D_1 \sin \rho y + D_2 \cos \rho y) dy$$

$$- \sum (\overline{K} + O(1/n))(\overline{C}_1 \sin \rho x + \overline{C}_2 \cos \rho x) \int_0^1 \phi(y)(\overline{D}_1 \sin \rho y + \overline{D}_2 \cos \rho y) dy$$

where both sums are extended over the same set of values of ρ. On substituting the evaluations of the integrals as given in Lemma X, we find, for values of n in the sequence $\{n_q\}$ of that lemma,

$$I_{n_q+1} - I_{n_q} = \sum \left(((KC_1D_1 - \overline{K}\overline{C}_1\overline{D}_1)n_q/\log n_q + o(n_q/\log n_q)) \sin \rho x \right.$$

$$\left. + ((KC_2D_1 - \overline{K}\overline{C}_2\overline{D}_1)n_q/\log n_q + o(n_q/\log n_q)) \cos \rho x \right)$$

where the sum is extended over a limited number of values of ρ of the form $\rho = 2n\pi + 2m_1\pi + a_1$, $\rho = 2n\pi + 2m_2\pi + a_2$. This expression cannot be $o(1)$ unless the coefficients of the terms in $n_q/\log n_q$ vanish. Thus we find the following three necessary conditions:

$$\cos \rho = \cos \bar{\rho}, \quad KC_1D_1 = \overline{K}\overline{C}_1\overline{D}_1, \quad KC_2D_1 = \overline{K}\overline{C}_2\overline{D}_1.$$

These three equations can be expressed in terms of the constants of the boundary conditions and are found to imply the equations

$$\bar{\alpha}_1 = \lambda\alpha_1, \quad \bar{\beta}_1 = \lambda\beta_1, \quad \bar{\alpha}_2 = \mu\alpha_2, \quad \bar{\beta}_2 = \mu\beta_2,$$

where λ and μ are constants different from zero.

When all the results are collected, the necessary condition enunciated in the theorem is seen to summarize them.

In order to demonstrate the sufficiency of the condition, as asserted in the last part of the theorem, we need only notice that when the boundary

conditions of the two systems fall into one of the three forms described the functions ϕ and $\bar\phi$ of Theorem III are identical. It follows that

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{\Gamma_n} 2\rho(G - \overline{G})d\rho\, dy = o(1)$$

uniformly, $0 < a \le x \le b < 1$, by a direct application of Theorem III.

In closing we may note that Lemma IX and, therefore, Theorem V may be extended to the case where the expressions considered are $o(1)$, not on a fixed interval, but only on a fixed point set of positive Lebesgue measure. The proof of the necessary condition of Lemma IX requires modification only at the point where an integral over the interval (a, b) is formed. Since Theorem V can be made just as general as the lemma on which it is based, the desired result is established.

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